# Variance Stabilizing Transformations

Suppose you have a random variable with the following mean and variance:

$$E[Y] = \mu \qquad V[Y] = \sigma^2 = \Omega(\mu)$$

We want a transformation $f(Y)$ that has constant variance. Writing out a first-order Taylor series expansion:

$$f(Y) \approx f(\mu) + (Y - \mu)f'(\mu)$$

$$\Rightarrow \qquad f(Y) - f(\mu) \approx (Y - \mu)f'(\mu)$$

$$\Rightarrow \qquad [f(Y) - f(\mu)]^2 \approx (Y - \mu)^2 (f'(\mu))^2$$

Now taking expectations on both sides, we get (again this is an approximation):

$$\Rightarrow \qquad V[f(Y)] \approx V(Y)[f'(\mu)]^2 = \Omega(\mu)[f'(\mu)]^2$$

Now, let:

$$f(\mu) = \int \frac{1}{[\Omega(\mu)]^{1/2}} d\mu$$

$$\Rightarrow \qquad V[f(Y)] \approx \left[\frac{\partial}{\partial\mu} \int [\Omega(\mu)]^{1/2} \left[\frac{1}{[\Omega(\mu)]}\right]^{1/2} d\mu\right]^2 = \left[\frac{\partial}{\partial\mu} \int d\mu\right]^2 = \left(\frac{\partial}{\partial\mu}(\mu + c)\right)^2 = 1$$

Thus, taking this transformation on $Y$ gives a random variable with an approximately constant variance.

**Example:** Suppose $\sigma^2 = \alpha^2 \mu^{2\beta} = \Omega(\mu)$

**Case 1:** $\beta \neq 1$

$$f(\mu) = \int \frac{1}{[\Omega(\mu)]^{1/2}} d\mu = \int \frac{1}{\alpha\mu^\beta} d\mu$$

$$\Rightarrow \qquad f(\mu) = \frac{1}{\alpha}\left[\frac{\mu^{-\beta+1}}{-\beta+1}\right] = c\mu^{1-\beta}$$

**Case 2:** $\beta = 1$

$$f(\mu) = \int \frac{1}{[\Omega(\mu)]^{1/2}} d\mu = \int \frac{1}{\alpha\mu} d\mu$$

$$\Rightarrow \qquad f(\mu) = \frac{1}{\alpha} log(\mu)$$

Now to estimate $\beta$:

$$\sigma = \alpha\mu^\beta \quad \Rightarrow \quad log(\sigma) = log(\alpha) + \beta log(\mu)$$

Thus, we can estimate $\beta$ by regressing $log(s_i)$ on $log(\overline{y}_{i.})$.

See example in Kuehl (p. 137), for this, where the logarithmic transformation is suggested. ($\hat{\beta} = 0.99$).