# Chapter 6 - Premium Calculations

## Section 6.2 - Preliminaries

To have an insurance benefit available, a policy holder must pay the insurance provider a premium or begin paying a series of premium payments.

If the policy is purchased via one payment at policy initiation, then the payment scheme is said to be a single premium. If, on the other hand, periodic payments are made over time, it is called a discrete contingent payment plan. Discrete refers to the periodic nature of the payments (annual, semi-annual, monthly, or from each paycheck). Contingent refers to the fact that these payments continue as long as the policy holder survives or (sometimes) until the policy holder reaches a certain age. Discrete contingent payments are typically level (the same amount is paid at each payment), but that is not a necessity. If payments differ, there is a payment scheme describing the payment progression set forth in the policy at the time of initiation. Premium payments always begin in advance of the insurance coverage.

When purchasing a life contingent annuity, if the annuity benefit payments begin immediately, then it is purchased with a single premium payment at the time of policy initiation. If the annuity benefit payments are deferred, then a discrete contingent payment plan could also be used to fund the annuity.

If the premium is set without specifically allowing for the insurance company's expenses, it is called a net premium (risk premium or mathematical premium). If the premium specifically includes company expenses, it is called a gross premium (office premium).

Example 6-1 A $100,000 whole life policy is issued to a person who is 2-year select at age [35]. The benefit is paid at the end of the year of the person's death. Premiums are paid annually beginning at policy initiation and are paid every year as long as the person survives until the person reaches age 65. If the policy holder survives beyond age 65, the policy remains in effect as

## Section 6.3 - Structural Assumptions

When determining a premium (pricing) for a policy, several ingredients are needed, one of which is

### Future Lifetime Distribution

We must have an anticipated future lifetime distribution that is appropriate for this individual. This is specified through a life table (typically a select life table), although continuous models are sometimes used for illustration. Our textbook specifies a standard select survival model life table for use in illustrating computations. It is based on a specific Makeham survival model and is displayed in Tables D.1, D.2 and D.3. Tables D.1 and D.2 are 2-year select tables and D.3 is an ultimate table. Tables D.2 and D.3 use $i = .05$.

## Section 6.4 - Loss Functions

A loss function includes the present value of the future benefits paid by the company and the present value of future premiums paid by the policy holder to the company.

When company expenses are not included it is described as a net future loss function:

If the PV of the benefits exceeds the PV of the premiums then the company loses money and $L_0^n > 0$. If the PV of the benefits is smaller than the PV of the premiums then the company makes money and $L_0^n < 0$. One or both of these PV's are random variables that depend on the future life length of the policy holder, $T_x$, which is unknown. So the value of $L_0^n$ is also a random variable.

The loss function sometimes includes company expenses, in which case it is called a gross future loss function:

Example 6-1 revisited:

The company expenses will also depend on the random curtate future life length, $K_x$.

# Section 6.5 - Premium Principle

This is the specified probabilistic plan whereby the premium value is determined. We will explore several such plans, the first and probably the most important of which is:

## Equivalence Principle

Under the equivalence principle the premium is determined so that the expected value of the future loss function (net or gross) is equal to zero. That is,

This implies that

Example 6-1 revisited: Use the net loss function,

$$L_0^n = (100,000)\nu^{K_{[35]}+1} - P\,\ddot{a}_{\overline{min(K_{[35]}+1,30)}|}.$$

Taking an expected value produces

$$E[L_0^n] = (100,000)A_{[35]} - P\,\ddot{a}_{[35]:\overline{30}|}.$$

Utilizing the equivalence principle, we set this equal to zero producing

$$P = \frac{(100,000)A_{[35]}}{\ddot{a}_{[35]:\overline{30}|}}.$$

We now use $i = .05$ and the tables in the back of the book to solve for P.

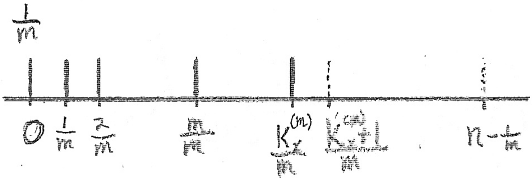
Example 6-2: Generalizing the previous example, suppose the whole life policy pays a benefit of $S$ at the end of the year of death, that it is set in force to someone at select age $[x]$ who pays annual premiums of $P$ for at most $n$ payments. Using the equivalence principle on the net loss function produces:

$$P = \frac{S\,A_{[x]}}{\ddot{a}_{[x]:\overline{n}|}}$$

Here $i$, the mortality distribution and the select period are all involved in the computation of $A_{[x]}$ and $\ddot{a}_{[x]:\overline{n}|}$.

Example 6-3 : Suppose a person age [x] qualifies for a deferred lifetime annuity. The person makes monthly premium payments of P during a deferral period of *n* years. At the end of the deferral period the annuitant begins paying the annuitant S per month contingent on the annuitant being alive at the time of payment. If the policyholder dies during the deferral period a lump sum death benefit of $S_*$ is paid at the end of the year of death. We seek to determine an appropriate value for *P* given *S*, $S_*$, a future life length distribution and *i*, using the equivalence principle.

Solution: First examine the EPV of premiums:



Here death occurs after surviving $K_x^{(m)}$ periods of length $\frac{1}{m}$ but before the end of the $(K_x^{(m)} + 1)^{th}$ period. Payments of $\frac{1}{m}$ (total of 1 per year) are life contingent with a maximum of $mn$ payments (stops after n years). This produces a EPV of

$$\ddot{a}^{(m)}_{[x]:\overline{n}|}.$$

So the EPV of premiums in the above setting is

The benefits are comprised of two ingredients. The first is a n-year term life insurance policy paying $S_*$ at the end of the year of death. This has a EPV of:

The second ingredient is a n-year deferred lifetime annuity with monthly payments of $S$ ( 12S paid annually) having a EPV of:

Thus by applying the equivalence principle to the net loss function, we get

In particular, using

$$[x] = [40] \qquad n = 25 \qquad i = .05 \qquad S_* = 25K \qquad S = 5K$$

and the standard select life tables of our textbook, we get:

Example 6-4: A luxury sports car is purchased for 40K. The theft probabilities for each year are shown in the table below. A three year theft policy is purchased at an annual net level premium of P. If a claim is payable at the end of the year of theft and $i = .10$, find P.

| $x$ | $q_x$ |
|---|---|
| 0 | .15 |
| 1 | .10 |
| 2 | .08 |
| 3 | .03 |

Example 6-5: For (0) you are given ${}_{k|}q_0 = \frac{1}{3}$ for k = 0, 1, 2. A policy pays 1 at the end of the year of death. The insured pays a net annual premium of P at the beginning of each year as long as the insured is alive. Using $i = .05$, find P.

## Continuous Payment Settings

While discrete premium payment schemes and discrete benefit payment settings are the most realistic, exam problems often use continuous payment settings as approximations and for mathematical convenience. Benefit schemes sometimes assume the benefit is paid at the moment of death. This means that they are using a continuous model for the future life length and the computation of the EPV of this benefit. Likewise, premium payments can be described as though they are paid continuously when they are paid over time. When the benefit is paid at the moment of death and the premiums are paid continuously, the insurance is said to be fully continuous. The equivalence principle is still applicable in these settings, with the premium being that value that makes the EPV of the benefits paid equal to the EPV of the premiums received.

Example 6-6: A person age 30 purchases a 20-year endowment policy of 100K with fully continuous payments and benefit paid immediately at death or at the end of 20 years whichever comes first. The constant force of mortality is $\delta = .06$. The net single premium for a 20-year endowment paying 1 is .125 . Find the net continuous annual premium.

Example 6-7 : Find the net premium for a fully continuous whole life insurance paying a benefit of $t$ immediately at death and having level continuous premiums if $\mu_{x+t} = \mu$ and $\delta_{x+t} = \delta$ for all $t > 0$.

## Section 6.6 - Gross Premiums (Including Expenses)

The premiums we have computed so far are called net premiums because they are based on the net loss function. When expenses are included to create a gross loss function and the equivalence principle is applied, they yield a gross premium. The gross loss function most often takes as additive form, so when using the equivalence principle to set the premium, the gross premium must satisfy:

### Types of Expenses

**Initial Expenses** - There are costs associated with securing the customer and underwriting the policy. These are often expressed as a linear function of the first year's premiums. The sales commission for life insurance, for example, is often a high percentage of the first year's premiums plus a much lower percentage of future premiums, payable as premiums are paid over time.

Renewal Expenses - These are expenses associated with billing, processing payments and periodic communications with policyholders. These expenses are often described on a per transaction (premium collection or benefit payment) basis, with the cost possibly growing by a fixed percentage over time to account for potential inflation.

Termination Expenses - These are the costs associated with the paperwork processing necessary when the policy ends. Typically these are relatively small and are often omitted in the present value of expenses computation.

Because insurance agents are usually paid a high(er) percentage of the first premium, that premium may not be sufficient to cover all of the initial expenses. Those up front expenses must be recouped over time. This results in a new business strain to cover the expenses of a rapidly expanding clientele.

Example 6-8 (Extending Example 6-2): Suppose the expenses incurred in writing this whole life policy are:

(a) \$1,000 up front cost

(b) 50% of the first premium

(c) 5% of all subsequent premiums

(d) \$100 annual renewal expense that increasing by 2% each year.

We seek to find the price, P .

The EPV of expenses is:

Example 6-9 (Extending Example 6-3): Find P when the expenses for the deferred lifetime annuity are:

(a) \$1,500 up front cost

(b) 15% of premiums paid during the first year of deferral

(c) \$3 per payment for premium processing, increasing by a factor of .15% per payment

(d) \$2 per benefit payment, increasing by a factor of .15% per payment period

The EPV of expenses is thus:

Example 6-10: Suppose $\mu_{x+t} = .025$ and $\delta_{x+t} = .05$ for all $t > 0$. A 25-year endowment policy pays a benefit of 200K at death or at 25 years, whichever comes first. Premiums are paid continuously and stop at death or at 25 years, whichever comes first. Expenses are \$1K plus .5% of premiums at payment. Find the annual gross premium using the equivalence principle.

Example 6-11: A company issues a three-year term insurance to a person whose age is 27. The insurance has a benefit of \$10,000 paid at the end of the year of death. Level premiums of P are paid at the beginning of each year as long as the insured is alive. Expenses are \$500 at the beginning , \$200 if the benefit is paid at the end of the first year and increases by 2% each year thereafter. Assume $i = .04$ and find P when the life table is shown below.

| $x$ | $q_x$ |
|---|---|
| 27 | .01 |
| 28 | .02 |
| 29 | .025 |
| 30 | .03 |

# Section 6.7 - Including Profit

So far we have not included profit in the premium computations.

Example 6-12: Suppose a person age 65 uses the \$700,000 in her 401K to purchase a life annuity that will pay B dollars annually, paid continuously until the moment of death. Suppose her future lifetime has a constant force of mortality of $\mu = .075$ and a constant force of interest $\delta = .04$. The expenses of setting up this annuity are \$1,000 at the time of issue of the policy. Therefore:

$$\text{EPV(Benefits)} + \text{EPV(Expenses)} = B\,\bar{a}_{65} + 1000$$

Using the equivalence principle, we set this expression equal to 700,000 and solve for B. This produces

$$\text{B} = \$80,385$$

as the amount paid annually to the annuitant.

The company's loss function random variable in this example is:

$$L_g(T_x) = 1000 + (80,385)\,\overline{a}_{\overline{T_x}|} - 700,000.$$

Find the probability that the company makes a profit.

## Profit through Rate Adjustment

One common way to include profit into the pricing equation is to adjust the interest rate $i$ ($\delta$) or to adjust the future lifetime distribution in each case making it a less favorable setting for the insurance company, thus raising the price to the client. In a whole life policy setting, for example, we could assume a lower $\delta$ than what we believe is possible and a higher mortality rate for the client. In an annuity setting we could assume a lower $\delta$ and a lower mortality rate, both being less favorable to the company.

Example 6-12 (Continued): Suppose when pricing the benefit, we had used $\delta = .035$ and $\mu = .065$. Then

$$\text{EPV(Benefits)} + \text{EPV(Expenses)} = B\,\overline{a}_{65} + 1000$$

When this is set equal to 700,000 it produces

$B_2 = \$69,000$ as the annual payment to the annuitant

The company's loss function random variable is based on the more realistic $\delta = .04$ and $\mu = .075$ and takes the form:

$$L_g(T_x) = 1000 + (69,000)\,\bar{a}_{\overline{T_x}|} - 700,000.$$

Find the probability that the company makes a profit with this structure.

Example 6-13: Consider a whole life insurance policy issued to someone age 20, with $T_{20}$ being Uniform (0,80). the policy pays 100K at the moment of death. It collects premiums continuously until the moment of death. Suppose, $\delta = .05$ and expenses are \$1,500 at issuance of the policy.

(a) Set up the gross loss function as a function of $T_{20}$.

(b) Determine the price P via the equivalence principle.

(c) Find $E[L_g(T_x)]$, if $\delta$ is actually .08.

(d) Find the probability of a profit when $\delta = .08$ .

# Section 6.8 - Portfolio Percentile Premium Principle

The second general principle for setting premium rates considers each policy to be embedded in a large portfolio of identical policies. The loss function random variable for the $i^{th}$ policy is:

$$L_{0i} \text{ for } i = 1, 2, \cdots, N$$

where $N$ is the number of identical policies in the portfolio. Moreover,

$L_1$, $L_2, \cdots, L_N$ are assumed to be i.i.d. ( independent & identically distributed) R.V.'s

The total portfolio loss random variable is thus

$$L = \sum_{i=1}^{N} L_{0i}.$$

It follows that

$$E[L] = \sum_{i=1}^{N} E[L_{0i}] = N\,E[L_{01}] \qquad \text{by identical dists..}$$

Also,

The Central Limit Theorem says that

has an approximate N( 0 , 1 ) distribution when N is large.

The portfolio makes a profit whenever $L < 0$ . Examine the probability that the portfolio makes a profit.

$$P[L < 0] \;=\; P\Big[L - N\,E[L_{01}] \;<\; -N\,E[L_{01}]\Big]$$

$$= P\left[\frac{L \;-\; N\,E[L_{01}]}{\sqrt{N\,Var[L_{01}]}} \;<\; \frac{-\,N\,E[L_{01}]}{\sqrt{N\,Var[L_{01}]}}\right]$$

where $\Phi(t)$ is the distribution function of a N( 0 , 1 ) distribution.

The Portfolio Percentile Premium Principle determines the premium so that the probability of making a profit is controlled and is set equal to $\alpha$ (a relatively high probability).

That is

$$\Phi\left(\frac{-N\,E[L_{01}]}{\sqrt{N\,Var[L_{01}]}}\right)=\alpha \qquad \text{or}$$

The term

$$\frac{E[L_{01}]}{\sqrt{Var[L_{01}]}}$$

will depend on the premium, which must be chosen to satisfy equation (6.8.1). The solution will clearly depend on the choices of $\alpha$ and $N$, Of course, $N$ must be sufficiently large to invoke the Central Limit Theorem (typically 30 or more).

Example 6-12 (Continued) : Take $\alpha = .90$ and $N = 100$. In this setting we are trying to determine the benefit amount B. With $\mu = .075$ and $\delta = .04$, the loss function is:

$$
\begin{aligned}
L_{01} &= 1,000 + B\, E[\bar{a}_{\overline{T_x}|}] - 700,000 \\
&= B \left[ \frac{1 - e^{-\delta T_x}}{\delta} \right] - 699,000.
\end{aligned}
$$

It follows that the expected value is

$$
\begin{aligned}
E[L_{01}] &= B\, E[\bar{a}_{\overline{T_x}|}] - 699,000 \\
&= B \left( \frac{1}{.075 + .04} \right) - 699,000 \;=\; B\,(8.695652) - 699,000.
\end{aligned}
$$

Likewise the

$$
Var[T_{01}] =
$$

# Section 6.9 - Extra Risks

Underwriting will reveal some applicants whose medical conditions, occupation or recreation choices put them in a higher risk category (Preferred $\rightarrow$ Normal $\rightarrow$ Rated) for life insurance than the typical insurance client. We have used "select" tables for individuals who are in a more favorable risk category, how might we adjust those tables for those who are less favorable with respect to risk?

## Subsection 6.9.1: Age Rating

The term impaired life is sometimes used to describe someone in a higher risk category. Because mortality probabilities $q_x$ increase with age (x) in human populations, one easy way to price a life insurance policy for such a person is to price the policy as though the person is age $x + k$ instead of their actual age $x$. The choice of $k$ being made on the basis of how many years their impairment subtracts from their life expectancy. This practice is referred to as age rating.

## Subsection 6.9.2: Constant Addition to the Force of Mortality

A second method used to adjust pricing computations to account for an increased risk of death is to add a positive constant to the force of mortality. Recall

Gompertz Model for FOM: $\mu_x = B\,C^x$

Makeham Model for FOM: $\mu_x = A + B\,C^x$.

Replacing

implies that the newly revised survival function becomes:

$$ {}_t p'_{[x]} = e^{-\int_0^t \mu'_{[x]+s} ds} = e^{-\int_0^t (\mu_{[x]+s}+\phi) ds} $$

Thus the net effect of adding a positive constant to the FOM is to decrease the survival function by a factor of $e^{-\phi t}$.

When computing the effect of this adjusted FOM on an annuity, for example, we see that

$$ \ddot{a}'_{[x]} = \sum_{k=0}^{\infty} \nu^k \, {}_k p'_{[x]} = \sum_{k=0}^{\infty} e^{-\delta k} \, e^{-\phi k} \, {}_k p_{[x]} $$

$$ = \sum_{k=0}^{\infty} e^{-(\delta+\phi)k} \, {}_k p_{[x]} = \ddot{a}_{[x]\,\delta'}. $$

The net effect here being that the force of interest is changed to $\delta' = \delta + \phi$.

However, the alteration of a single benefit payment is not as simple, for example,

$$\begin{aligned}
\overline{A}'_{[x]} &= E\left[e^{-\delta\, T_x}\right] = \int_0^\infty e^{-\delta t} \mu'_{[x]+t}\, {}_t p'_{[x]} dt \\
&= \int_0^\infty e^{-\delta t} (\mu_{[x]+t} + \phi)\, E^{-\phi t}\, {}_t p_{[x]} dt \\
&= \int_0^\infty e^{-\delta' t}\, \mu_{[x]+t}\, {}_t p_{[x]} dt \, + \, \phi \int_0^\infty e^{-\delta' t}\, {}_t p_{[x]} dt
\end{aligned}$$

Thus both $\overline{A}$ and $\overline{a}$ must be converted to $\delta'$ to compute $\overline{A}'$. It is often easier to note that

$$\begin{aligned}
\overline{a}_{[x]\,\delta'} &= \overline{a}'_{[x]} = E\left[\overline{a}_{\overline{T'_{[x]}}|}\right] \\
&= \frac{1 - E\left[\nu^{T'_{[x]}}\right]}{\delta} = \frac{1 - \overline{A}'_{[x]}}{\delta}.
\end{aligned}$$

Therefore,

This shows that the new $\overline{A}'$ values can be determined by $\overline{a}$ values computed with $\delta'$ .

## Subsection 6.9.3: Constant multiple of Mortality Rates

A life table can be constructed from the $q_{[x]+t}$ (mortality rate) column. The $d_x$ column comes from these values and the $l_x$ does also. By increasing all the $q_{[x]+t}$ vlues by a fixed ercentage, a situation with a higher rate of morbidity is modelled. We form:

$$q'_{[x]+t} = q_{[x]+t}(1+j)$$

with $j > 0$ and a new more appropriate life table is then constructed for an impaired client.

## Example 6-14:

| $x$ | $q_x$ | $l_x$ | $d_x$ | $q'_x$ | $l'_x$ | $d'_x$ |
|---|---|---|---|---|---|---|
| 55 | .006275 | 91497 | 574 | .006905 | 91497 | 632 |
| 56 | .006726 | 90923 | 612 | .007399 | 90865 | 672 |
| 57 | .007220 | 90311 | 652 | .007942 | 90193 | 716 |
| 58 | .007773 | 89659 | 697 | .008550 | 89477 | 765 |
| 59 | .008389 | 88962 | 746 | .009228 | 88712 | 819 |
| 60 | .009081 | 88216 | 801 | .009989 | 87893 | 878 |

The left-hand part of this table was taken directly from the UD Life Tables in Chapter 3. The right-hand side was formed by multiplying $q_x$ by (1.1), a 10% increase, creating $q'_x$. The value $l_{55} = l'_{55}$. Subsequent values on the right-hand side were found by forming

$$d'_x = \lfloor q'_x \, l'_x \rfloor \qquad \text{and} \qquad l'_x = l'_{x-1} - d'_{x-1}.$$

Use the original table to find the annual premium of a \$50,000 5-year term life insurance policy (55) with annual premium payments. Suppose expenses are \$500 plus 50% of the first premium. Use $i = .05$ and the equivalence principle.

Now use the impaired life table to price the same policy.

Example 6-15: Suppose the life distribution is $T_0 \sim$ deMoivre (0,100) and a \$200,000 policy is whole life paid immediately at death for someone (40). Premiums are paid continuously until death. Use $\delta = .04$.

(a) Find the net premium.

(b) Suppose this policy holder is age rated 5 years, find the net premium under this assumption.