Chapter 2 - Survival Models

Section 2.2 - Future Lifetime Random Variable and the Survival Function

Let

$$T_x = (\text{ Future lifelength beyond age } x \text{ of an individual who has}$$
$$\text{survived to age } x \text{ [measured in years and partial years]})$$

The total lifelength of this individual will be $x + T_x$, i.e. this is the age at which the individual dies [including partial years].

The additional years of life T_x beyond x is unknown and therefore is viewed as a continuous random variable. The distribution of this random variable is described by or by

where, of course,

$$F_x(t) = \int_0^t f_x(s) ds.$$

Either of the functions $f_x(t)$ or $F_x(t)$ are used to describe the future lifetime distribution beyond age x. Clearly, $F_x(t) = P[T_x \leq t]$ is the probability that someone who has survived to age x will not survive beyond age $x + t$. Therefore,

is the probability that someone age x does survive t additional years. All of the properties of the future lifetime distribution are in the survival function $S_x(t)$.

Properties of a Survival Function $S_x(t)$

Property 1:

$$S_x(0) = 1.$$

Everyone who survived to age x is alive at the beginning of the time period beyond x.

Property 2:

No one lives infinitly long beyond x.

Property 3: If $t_1 < t_2$ then

$$S_x(t_1) \geq S_x(t_2).$$

The function $S_x(t)$ is non-increasing.

Let T_0 denote the total lifelength from birth of an arbitrary individual. The density of its distribution is $f_0(t)$.
Note that

$$F_x(t) = P[T_x \leq t] = P[x < T_0 \leq x + t \mid T_0 > x]$$

$$= \frac{P[x < T_0 \leq x + t]}{p[T_0 > x]} = \tag{2.1}$$

Taking a derivative with respect to t produces

So the $f_x(\cdot)$ density is proportional to the $f_0(\cdot)$ density at the corresponding time point.

From expression (2.1) we also see that

$$F_x(t) = \frac{S_0(x) - S_0(x+t)}{S_0(x)} = 1 - \frac{S_0(x+t)}{S_0(x)}$$

Therefore

which is the fraction alive at x who continue to be alive at $x+t$.

Rewriting this expression produces

$$S_0(x+t) = S_0(x)S_x(t)$$

which shows that the probability of surviving $x + t$ years is the probability of surviving x years times the conditional probability of surviving t additional years given survival to time x.

More generally, the same reasoning produces

which shows that the probability of surviving $t + u$ years beyond x is the probability of surviving t years beyond x times the conditional probability of surviving u additional years given survival to time $x + t$.

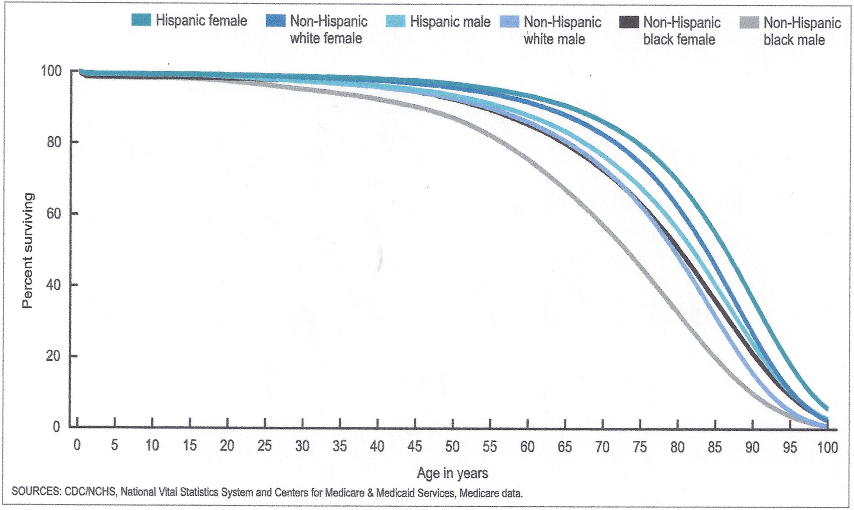


Figure 3. Percentage surviving, by Hispanic origin, race, age, and sex: United States, 2007

Assumptions for a Survival Function $S_x(t)$ that are useful when finding expected values

Assumption 1:

The survival function $S_x(t)$ is a smooth nonincreasing function of t.

Assumption 2:

$$\lim_{t \to \infty} t S_x(t) = 0.$$

The right-hand tail of the survival function goes to zero sufficiently fast as t goes to infinity.

Assumption 3:

$$\lim_{t \to \infty} t^2 S_x(t) = 0.$$

The right-hand tail of the survival function goes to zero even faster as t goes to infinity.

Example 2-1: Let ω denote some upper age limit (e.g. 120) and

$$f_0(t) = \begin{cases} \frac{12}{\omega}\left(\frac{t}{\omega}\right)^2\left(1 - \frac{t}{\omega}\right) & \text{for } 0 < t < \omega \\ 0 & \textit{elsewhere} \end{cases}$$

Find $F_0(t)$, $S_x(t)$ for general ω and $S_{40}(10)$ when $\omega = 120$.

Section 2.3 - Force of Mortality

Concept - At any age, what is the rate of death among persons who have survived to that age?

Large positive number $\longrightarrow$ hazardous age
Small positive number $\longrightarrow$ less hazardous age

Define the Force of Mortality at age x to be

$$\mu_x = \lim_{dx \searrow 0} \frac{P[x < T_0 < x + dx \mid T_0 > x]}{dx}$$

$$= \frac{\lim_{dx \searrow 0} \frac{F_0(x+dx) - F_0(x)}{dx}}{S_0(x)} \qquad \text{or}$$

Force of Mortality is a function of the age x of the individual. It is also called the hazard function or the failure rate function. Note that

$$\mu_x = \frac{F_0'(x)|_{t=x}}{S_0(x)}$$

$$= \frac{\frac{d}{dt}\Big(1 - S_0(t)\Big)\Big|_{t=x}}{S_0(x)} \qquad \text{or}$$

This shows that the survival function characterizes the force of mortality. Note also that

$$\mu_x = \frac{d}{dx}\Big[- ln(S_0(x))\Big] \qquad \text{so}$$

$$\int_0^t \mu_x dx = -\ln(S_0(t)) + \ln(S_0(0)).$$

It follows that

Therefore the force of mortality function characterizes the survival function.

Note also that

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{e^{-\int_0^{x+t} \mu_r dr}}{e^{-\int_0^{x} \mu_r dr}}$$

$$= e^{-\int_x^{x+t} \mu_r dr} = e^{-\int_0^{t} \mu_{x+r} dr}$$

In the same manner we see

$$\mu_{x+t} = \frac{-S_0'(x+t)}{S_0(x+t)}$$

$$= \frac{-\lim_{\Delta \searrow 0} \left(\frac{S_0(x+t+\Delta) - S_0(x+t)}{\Delta} \right)}{S_0(x+t)}$$

$$= \frac{-\lim_{\Delta \searrow 0} \left(\frac{S_0(x)S_x(t+\Delta) - S_0(x)S_x(t)}{\Delta} \right)}{S_0(x+t)}$$

$$= \frac{-S_0(x)}{S_0(x+t)} \lim_{\Delta \searrow 0} \frac{S_x(t+\Delta) - S_x(t)}{\Delta} \qquad \text{or}$$

Example 2-2: Given $\mu_x = \frac{1}{100-x}$ for $0 < x < 100$,

find $S_{50}(10) = P[T_{50} > 10]$.

Example 2-3: Given $\mu_x = 2x$ for $0 < x$,

find $f_0(t), F_0(t), S_0(t)$ and $f_x(t)$.

Gompertz Law of Mortality (1825):

where $0 < B < 1$ and $C > 1$.

Here the force of mortality is increasing exponentially. It follows that the survival function is:

$$
\begin{aligned}
S_x(t) &= e^{-\int_0^t \mu_{x+r} dr} \\
&= e^{-BC^x \int_0^t C^r dr} \\
&= e^{-BC^x \left[\frac{C^r}{ln(C)}\right]_0^t} \\
&= e^{-\frac{BC^x}{ln(C)}\left(C^t - 1\right)}
\end{aligned}
$$

Makeham Law of Mortality (1860):

where $A > 0, 0 < B < 1$ and $C > 1$.

The coefficient B is part of what determines the rate of ascent of the force of mortality. It is also part of the value of the force of mortality when $x = 0$. The addition of the coefficient A allows an adjustment to the force of mortality at $x = 0$ that is not part of its rate of ascent. The survival function is now:

$$S_x(t) = e^{-A\int_0^t dr - BC^x \int_0^t C^r dr}$$

$$= e^{-tA - \frac{BC^x}{ln(C)}\left(C^t - 1\right)}$$

Section 2.4 - Actuarial Notation

Having survived to age x, the probability of surviving t additional years is:

Having survived to age x, the probability of NOT surviving t additional years is:

Having survived to age x, the probability of surviving u additional years and then dying within t years after $x + u$, is:

$$_{u|t}q_x = S_x(u) - S_x(u+t) = P[u < T_x < u+t]$$

This is referred to as a deferred mortality (here deferred u years).

It follows that

$$ {}_{u|t}q_x = {}_u p_x - {}_{u+t}p_x = {}_u p_x ({}_t q_{x+u}). $$

Also,

Note that

$$ \mu_x = -\frac{S_0'(x)}{S_0(x)} = \frac{-\frac{d}{dx}({}_x p_0)}{{}_x p_0}. $$

In the same manner,

But since

$$\frac{d}{dt}{}_tp_x = \frac{d}{dt}S_x(t) = -f_x(t),$$

we also get

Using the material from section 2.2, we see that

Likewise, we have

Section 2.5 - Properties of T_x

The future lifetime at age x, T_x, is a continuous random variable. We are interested in the properties of this random variable. In particular, its mean is called the complete expectation of life and is equal to

$$= \int_0^\infty t\,({}_tp_x)\mu_{x+t}dt$$

$$= -\int_0^\infty t\Big(\frac{d}{dt}{}_tp_x\Big)dt$$

$$= -t\,({}_tp_x)\Big|_0^\infty + \int_0^\infty {}_tp_x dt,$$

producing the computation formula

In a similar manner, the computation formula for the second moment of T_x is

It follows that the variance of T_x is computed via

$$Var[T_x] = E[T_x^2] - (\mathring{e}_x)^2.$$

and, of course the standard deviation of T_x is

$$StD[T_x] = \sqrt{Var[T_x]}.$$

The percentiles of the distribution of T_x are of interest.

In particular, the Median, m(x), (the 50th percentile) is the value which satisfies

Another concept is:

$$\overset{\circ}{e}_{x:\overline{n}|} \equiv \text{Average number of years lived within the next n years}$$

It can be computed with

$$\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n t\, f_x(t) dt + nP[T_x > n].$$

The Central Death Rate

$$ {}_t m_x = \frac{\int_0^t \mu_{x+s}\, {}_s p_x ds}{\int_0^t {}_s p_x ds} $$

is a weighted average of the Force of Mortality values over the interval from x to $x + t$.

Example 2-4: Continuing example 2-1, find

$\mathring{e}_x$, StDev(T_x), and m(x).

Section 2.5.5 - Some Important Mortality Models

Uniform Distribution or DeMoivre's Law

$$f_0(t) = \begin{cases} \frac{1}{\omega} & \text{if } 0 < t < \omega \\ 0 & \text{elsewhere} \end{cases}$$

$$_tq_0 = F_0(t) = \frac{t}{\omega} \quad \text{for } 0 < t < \omega$$

$$_tp_0 = \frac{\omega - t}{\omega} \quad \text{for } 0 < t < \omega.$$

With this model, if we assume the person has already lived to age x, then

The force of mortality under DeMoivre's Law is

Note that it is an increasing function of the age x. That is, life is more hazardous as we get older under this model.

Note also that T_x is also uniform $(0, \omega - x)$ and thus, for example,

$$\mathring{e}_x = E[T_x] = \frac{\omega - x}{2}, \qquad m(x) = \frac{\omega - x}{2} \qquad \text{and}$$

$$Var[T_x] = \frac{(\omega - x)^2}{12}.$$

An important property of the DeMoivre Law (Uniform Distribution) is its reproducibility. If a future lifelength is uniform, then the future lifelength beyond any future age is also uniform. That is, if T_x is uniform $(0, \omega - x)$, then T_{x+y} is uniform $(0, \omega - x - y)$. So future life length distributions stay within the class of uniform distributions, it merely changes the parameter of the distribution (the length of the interval in this case).

Exponential Distribution

$$
\begin{aligned}
{}_tq_0 = F_0(t) &= \int_0^t \frac{1}{\theta} e^{-\frac{s}{\theta}} ds \\
&= -e^{-\frac{s}{\theta}} \Big|_0^t \\
&= 1 - e^{-\frac{t}{\theta}} \quad \text{for } 0 < t
\end{aligned}
$$

$$
{}_tp_0 = e^{-\frac{t}{\theta}} = S_0(t) \quad \text{for } 0 < t
$$

Now suppose this exponential function describes survival from birth and that the person has already lived to age $x > 0$. The density of future life length beyond x is

This clearly shows that the future life length beyond x has exactly the same distribution as the original life length from birth.

The exponential distribution has an even stronger reproducibility property than the uniform distribution had. Under the exponential distribution for future life length, the life length distribution beyond any point in the future is exactly the same exponential distribution that is applicable beyond today (same distribution AND the same parameter value).

For the exponential distribution:

$$\mathring{e}_x = E[T_x] = \int_0^\infty {}_t p_x dt = \int_0^\infty e^{-\frac{t}{\theta}} dt$$

$$E[T_x{}^2] = 2\int_0^\infty t({}_t p_x) dt = 2\int_0^\infty t e^{-\frac{t}{\theta}} dt$$

$$= 2\Big[-\theta t e^{-\frac{t}{\theta}}\Big|_0^\infty + \theta^2 \int_0^\infty \frac{1}{\theta} e^{-\frac{t}{\theta}} dt\Big]$$

$$= 2\theta^2.$$

Therefore

$$Var[T_x] = 2\theta^2 - \theta^2 = \theta^2 \qquad \text{and}$$

For the exponential, the force of mortality is

$$\mu_x = -\frac{d}{dt} S_x(t)\Big|_{t=0} = \frac{1}{\theta} e^{-\frac{t}{\theta}}\Big|_{t=0} = \frac{1}{\theta}.$$

Moreover, a constant force of mortality characterizes an exponential distribution. Let μ^* denote a constant force of mortality. Then

This is, of course, the survival function of an exponential distribution with

$$\mu^* = \frac{1}{\theta}.$$

While a constant force of mortality throughout life is unrealistic, MLC exam questions frequently assume different constant forces of mortality over various segments of a lifetime.

Weibull Distribution

This family of distributions has two parameters: a scale parameter $\theta > 0$ and a shape parameter $\tau > 0$. Its survival function takes the form:

This produces a density function of the form

$$f_0(t) = \begin{cases} \frac{\tau}{\theta}\left(\frac{t}{\theta}\right)^{\tau-1} e^{-\left(\frac{t}{\theta}\right)^{\tau}} & \text{for } 0 < t \\ 0 & \text{for } t \leq 0 \end{cases}$$

and a distribution function

$$ {}_tq_0 = F_0(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^{\tau}}$$

The Weibull force of mortality function is:

When $\tau > 1$, this is an increasing function of x (proper for mortality) though it does spread mortality over the whole positive part of the real line.

When $\tau < 1$, this is an decreasing function of x (generally improper for mortality).

When $\tau = 1$, the Weibull is the exponential distribution and is only appropriate for relatively short periods of time.

Using the Weibull distribution to describe mortality from birth, the future lifelength beyond age x satisfies

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{e^{-\left(\frac{t+x}{\theta}\right)^{\tau}}}{e^{-\left(\frac{x}{\theta}\right)^{\tau}}}$$

which is not a survival function of a Weibull distribution (it lacks reproducibility).

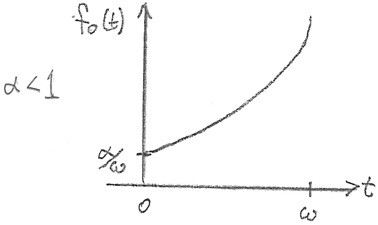
Also note that

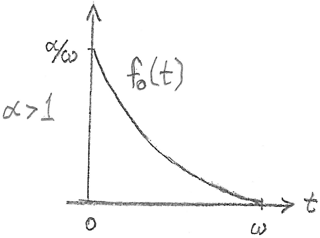
$$\overset{\circ}{e}_x = \theta\Gamma\left(\frac{\tau+1}{\tau}\right)$$ and

$$Var[T_x] = \theta^2\left\{\Gamma\left(\frac{\tau+2}{\tau}\right) - \left[\Gamma\left(\frac{\tau+1}{\tau}\right)\right]^2\right\}$$

Generalized DeMoivre (Beta)

Here ω, the maximum age, is essentially a scale parameter and α is a shape parameter.





When $\alpha = 1$, this is DeMoivre's Law, ie it is a uniform $(0, \omega)$ distribution. We also note that for the generalized DeMoivre distribution

$$ {}_tq_0 = F_0(t) = 1 - \left(\frac{\omega - t}{\omega}\right)^{\alpha} \qquad \text{for } 0 < t < \omega \qquad \text{and} $$

Suppose The generalized DeMoivre applies form birth, but the individual has survived to age $x > 0$. The density of the future lifelength beyond x is:

$$ f_x(t) = \frac{f_0(x+t)}{{}_xp_0} = \begin{cases} \frac{\alpha}{\omega - x}\left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha - 1} & \text{if } 0 < t < \omega - x \\ 0 & \text{elsewhere} \end{cases} $$

We see that this conditional distribution is also a member of the generalized DeMoivre family with scale parameter $\omega - x$ and the same shape parameter α. So this family has a reproducibility property.

The force of mortality function for the generalized DeMoivre is

This is a decreasing function of x for all $\alpha > 0$. Like the DeMoivre Law this generalized family is best applied to relatively short periods of time.

Also note that

$$\mathring{e}_x = \frac{\omega - x}{\alpha + 1} \qquad \text{and}$$

$$Var[T_x] = \frac{(\omega - x)^2 \alpha}{(\alpha + 1)^2 (\alpha + 2)}.$$

Example 2-5: You are given that there is a constant force of mortality μ^* and that $\mathring{e}_{30} = 41$. Find μ^*.

Example 2-6: You are given $S_0(t) = \left(1 - \frac{t}{\omega}\right)^{\alpha}$ for $0 < t < \omega$ and $\alpha > 0$. Derive $\overset{\circ}{e}_x$ and then find $\mu_x \overset{\circ}{e}_x$.

Section 2.6 - Curtate Future Lifetime

When describing a number of features of a policy, e.g. the number of future annual premium payments, it is useful to model the integer which represents the whole number of future years lived by a person who is currently age x. This is the discrete random variable

where $\lfloor t \rfloor$ is the largest integer that is less than or equal to t.

We note that

$$P[K_x = k] = P[\text{individual survives } k \text{ years but not } k+1 \text{ years}]$$

$$= P[k \leq T_x < k+1]$$

$$= {}_k p_x - {}_{k+1} p_x \quad = \quad {}_k p_x - {}_k p_x \, p_{x+k}$$

$$= {}_k p_x (1 - p_{x+k}) = \quad {}_k p_x \, q_{x+k}.$$

The expected value of K_x is denoted by e_x and can be computed via

$$e_x = E[K_x] = \sum_{k=0}^{\infty} k\, P[K_x = k]$$

$$= 1({}_1p_x - {}_2p_x) + 2({}_2p_x - {}_3p_x) + 3({}_3p_x - {}_4p_x) + \cdots$$

$$= {}_1p_x + {}_2p_x + {}_3p_x + \cdots$$

$$= \sum_{k=1}^{\infty} {}_kp_x$$

Likewise the second moment is:

$$E[K_x{}^2] = \sum_{k=0}^{\infty} k^2\, P[K_x = k]$$

$$= \sum_{k=0}^{\infty} k^2\, ({}_kp_x - {}_{k+1}p_x)$$

$$= 1^2({}_1p_x - {}_2p_x) + 2^2({}_2p_x - {}_3p_x) + 3^2({}_3p_x - {}_4p_x) + \cdots$$

$$= \sum_{k=1}^{\infty}(2k-1)\, {}_kp_x$$

$$= 2\sum_{k=1}^{\infty} k\, {}_kp_x - \sum_{k=1}^{\infty} {}_kp_x$$

$$= 2\sum_{k=1}^{\infty} k\, {}_kp_x - e_x.$$

Therefore,

Because

$$T_x \geq K_x > T_x - 1,$$

$$0 \leq T_x - K_x < 1.$$

As an approximation, it is sometimes assumed that in a short period of time (eg one year) deaths occur uniformly. Thus it is assumed

$$(T_x - K_x) \sim uniform(0,1) \quad \text{and therefore} \quad E[T_x - K_x] = \frac{1}{2}.$$

Based on this assumption

$$\mathring{e}_x = E[T_x] = E[K_x + (T_x - K_x)]$$

$$\doteq e_x + \frac{1}{2}.$$

Example 2-7: Suppose $T_0 \sim$ DeMoivre with $\omega = 100$. Find the curtate mean e_{20}.