Chapter 5: Summary

Chebyshev's Inequality:

Suppose $X$ is a random variable with $E(X) = \mu$ and $V(X) = \sigma^2$. Then for every $t > 0$,

$$P(|X - \mu| > t) \leq V(X)/t^2.$$

Law of Large Numbers:

Suppose $X_1, \cdots, X_n$ are independent with common mean $\mu$ and common variance $\sigma^2$. Let $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$. Then by Chebyshev's Inequality, for every $\epsilon > 0$,

$$P(|\bar{X}_n - \mu| > \epsilon) \leq V(\bar{X}_n)/\epsilon^2 = \frac{\sigma^2}{n\epsilon^2} \to 0 \text{ as } n \to \infty.$$

This implies that $\bar{X}_n$ converges in probability to $\mu$, written symbolically as $\bar{X}_n \xrightarrow{P} \mu$. This is the simplest version of the law of large numbers.

Central Limit Theorem:

Suppose $X_1, \cdots, X_n$ are i.i.d. with mean $\mu$ and variance $\sigma^2$ $(0 < \sigma^2 < \infty)$. Let $S_n = \sum_{i=1}^n X_i$, $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$. Then

$$P\left(\frac{n^{1/2}(\bar{X}_n - \mu)}{\sigma} \leq x\right) = P\left(\frac{S_n - n\mu}{n^{1/2}\sigma} \leq x\right) \to \Phi(x)$$

as $n \to \infty$, where $\Phi(x)$ is the cumulative distribution function of a N(0,1) random variable. We say that $\bar{X}_n$ is asymptotically N$(\mu, \sigma^2/n)$ or $S_n$ is asymptotically N$(n\mu, n\sigma^2)$.