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BOUNDS ON THE EXTINCTION TIME DISTRIBUTION OF A BRANCHING PROCESS

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Abstract

The class of fractional linear generating functions, one of the few known classes of probability generating functions whose iterates can be explicitly stated, is examined. The method of bounding a probability generating function g (satisfying $g''(1)<\infty$) by two fractional linear generating functions is used to derive bounds for the extinction time distribution of the Galton-Watson branching process with offspring probability distribution represented by g. For the special case of the Poisson probability generating function, the best possible bounding fractional linear generating functions are obtained, and the bounds for the expected time to extinction of the corresponding Poisson branching process are better than any previously published.

GALTON-WATSON PROCESS; EXTINCTION TIME; FRACTIONAL LINEAR GENERATING FUNCTION

1. Introduction

In this paper bounds are derived for the extinction time distribution of a one type Galton-Watson branching process. This process represents the evolution of a population in which different individuals reproduce independently of each other and the offspring probability distribution is identical for every individual and every generation. The Galton-Watson process has been used as a simplified model for such problems as determining the fate of the population generated by a newly mutant gene.

Suppose that the offspring probability distribution of each individual in the population is represented by the probability generating function (p.g.f.) $g(s) = \sum_{j=0}^{\infty} p_j s^j$, $0 \leqq s \leqq 1$, and let $m = g'(1)$, the mean number of offspring produced by each individual. To avoid trivialities, assume that $p_0 + p_1 < 1$ and $0 < p_0$. Letting Z_n denote the size of the population at the nth generation, it is well known that the p.g.f. for Z_n if $Z_0 = 1$ is $g_n(s) = g(g(\cdots g(s)\cdots))$, the n-fold iterate of g. However, there are very few families of p.g.f.'s whose iterates have a simple closed form expression. The approach used in this paper results in bounds for $\{g_n(s),\ 0 \leqq s \leqq 1,\ n \geqq 1\}$, and hence for $\{g_n(0) = P(Z_n = 0) = P(T \leqq n),\ n \geqq 1\}$, where T is the extinction time of the branching process. These bounds

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are used to obtain bounds for the percentiles of the distribution of T and other parameters of interest. As usual, subcritical, critical and supercritical processes refer to $m < 1$, $m = 1$ and $m > 1$, respectively. Without loss of generality, we shall assume throughout that $Z_0 = 1$.

Our main approach consists of reducing the problem of deriving bounds for the Galton-Watson process to a problem involving the one family of p.g.f.'s whose iterates can be easily calculated, the fractional linear generating functions. Seneta (1967) noted that if U and L are two p.g.f.'s such that

$$L(s) \leqq g(s) \leqq U(s), \quad 0 \leqq s \leqq 1,$$

then

$$L_n(s) \leqq g_n(s) \leqq U_n(s),\ 0 \leqq s \leqq 1,\ n \geqq 1.$$

As a result,

$$L_n(0) \leqq P(T \leqq n) \leqq U_n(0), \qquad n \geqq 1, \tag{1.1}$$

and when $m < 1$ the expected time to extinction $ET = \sum_{n=0}^{\infty} [1 - g_n(0)]$ is bounded by

$$\sum_{n=0}^{\infty} [1 - U_n(0)] \leqq ET \leqq \sum_{n=0}^{\infty} [1 - L_n(0)], \tag{1.2}$$

and when in addition $g''(1) < \infty$, $\mu = \lim_{n \to \infty} m^n/(1 - g_n(0))$ is bounded by

$$\lim_{n \to \infty} m^n/(1 - L_n(0)) \leqq \mu \leqq \lim_{n \to \infty} m^n/(1 - U_n(0)). \tag{1.3}$$

Seneta derived two fractional linear generating functions which bound a subcritical Poisson p.g.f., producing good bounds for μ and ET for a Poisson branching process. In Section 3, we find two fractional linear generating functions which bound any subcritical or critical p.g.f. g with $g''(1) < \infty$ in such a way that the means of the bounding functions are equal. In Lemmas 2 and 3 and in Theorem 2, we show that for some special cases (such as the Poisson p.g.f.) it is easy to find the best bounding fractional linear generating functions with mean m. Bounds are obtained for supercritical processes in Section 4 by exploiting a duality between subcritical and supercritical processes. Since for a large class of Galton-Watson processes the variance of T grows exponentially faster than ET as $m \to 1^-$ (see Seneta (1968)), and since these measures are infinite for the unconditioned process when $m \geqq 1$, we concentrate on obtaining good bounds for $\{P(T \leqq n),\ n \geqq 1\}$ and for the percentiles ξ_p of T. Bounds for ET and μ are easily expressed as a by-product of these bounds when $m < 1$.

The problem of obtaining bounds on the extinction time distribution of a Galton-Watson process has been considered recently in a few papers. Heathcote and Seneta (1966) presented bounds for ET and μ for subcritical processes with $g'''(1) < \infty$. Under very general conditions (e.g., see Seneta (1968)), as $m \to 1^-$

their lower bound for ET converges to a finite limit (although $ET \to \infty$), and their upper bound grows at a rate proportional to $(1-m)^{-1}$, which is exponentially faster than the actual rate. Pollak (1971) has also considered the problem of deriving bounds for ET and μ. His bounds apply to p.g.f.'s which can be shown to satisfy a certain inequality involving the first three derivatives of the p.g.f. evaluated at one. However, there is a large class of p.g.f.'s for which these bounds are inapplicable, and in general it is difficult to verify whether the bounds apply to a given p.g.f. Seneta (1967), as described above, and Pollak (1969) have also derived bounds for the extinction time distribution when g has the Poisson form, which has been used in genetic applications of branching processes since Fisher (1930). The bounds derived in this paper are applicable to *all* Galton-Watson processes with $g''(1) < \infty$, yet still maintain good properties for values of m close to one, which are of special interest in many applications. In particular, the bounds for ET for a Poisson branching process are better than any previously published.

2. The class of fractional linear generating functions

A fractional linear generating function (f.l.g.f.) is a p.g.f. of the form

$$(2.1) \qquad f(b,c;s) = 1 - \frac{b}{1-c} + \frac{bs}{1-cs}, \qquad 0 \leqq s \leqq 1,$$

where $0 \leqq b \leqq 1$, $0 \leqq c < 1$, and $b + c \leqq 1$. Corresponding to this p.g.f. is the two-parameter geometric distribution $\{r_j, j \geqq 0\}$, where $r_0 = 1 - b/(1-c)$ and $r_j = bc^{j-1}$, $j \geqq 1$. If $c = 0$, the distribution reduces to a Bernoulli trial with $r_0 = 1 - b$, $r_1 = b$, and $f(b,c;s)$ is linear. If $b = 1 - c$, it reduces to the geometric distribution $r_j = (1-c)c^{j-1}$, $j \geqq 1$. The mean of a two-parameter geometric distribution is $m = f'(b,c;1) = b(1-c)^{-2}$, and $f''(b,c;1) = 2bc(1-c)^{-3}$. If $m \neq 1$, there are two distinct non-negative solutions to the equation $f(b,c;s) = s$, 1 and $s_0 = s_0(b,c) = (1-b-c)/c(1-c)$. If $m < 1$, then $s_0 > 1$; if $m > 1$, then $s_0 < 1$ and $s_0 = q$, the probability of extinction of the branching process with p.g.f. $f(b,c;s)$; $s_0 = 1$ if and only if $m = 1$. Iterates of f.l.g.f.'s are also f.l.g.f.'s and can be explicitly stated. In Section 3, the extinction time distribution of a Galton-Watson process with f.l.g.f. is detailed, and then is exploited in Theorem 1. In the remainder of this section, we prove other properties of the class of f.l.g.f.'s which are used in this paper.

To use the bounding approach described in the previous section, it is necessary to determine which of the f.l.g.f.'s that bound a p.g.f. g produce the best bounds for the extinction time distribution. When $m \neq 1$, $q - g_n(0) \sim \mu^{-1}[g'(q)]^n$ as $n \to \infty$ (see Harris (1963), p. 16, 18), so that $\xi_p \sim \log \mu(q-p)/\log g'(q)$ as $p \to q^-$. In general, μ is unknown. Good asymptotic bounds for $P(T \leqq n)$ or for ξ_p are obtained by restricting attention to those f.l.g.f.'s that bound g in such a way that $f(b,c;q) = q$ and $f'(b,c;q) = g'(q)$. When $m = 1$ and $g''(1) < \infty$, Kolmogorov (see Harris (1963), p. 21) showed that $1 - g_n(0) \sim 2/g''(1)n$ as $n \to \infty$, so that $\xi_p \sim 2/g''(1)(1-p)$ as $p \to 1^-$. As a result of Lemma 1, we shall see that when $m \leqq 1$ (so that $g'(q) = m$), the best asymptotic bounds using this approach are produced by the two f.l.g.f.'s with mean m that bound g and have second derivative at one as close as possible to $g''(1)$, or alternatively, have r_0 as close as possible to p_0.

Lemma 1. For $f'(b,c;1) = m \leqq 1$ fixed:

(i) $f(b,c;s)$ is an increasing function of $f''(b,c;1)$ for all s, $0 \leqq s < 1$;
(ii) $f(b,c;s)$ is an increasing function of $r_0 = 1 - b/(1-c)$ for all s, $0 \leqq s < 1$.

Proof. Since $f'(b,c;1) = b(1-c)^{-2}$ and $f''(b,c;1) = 2bc(1-c)^{-3}$, we have

$$b = 4f'(b,c;1)^3[2f'(b,c;1) + f''(b,c;1)]^{-2}$$

and

$$c = f''(b,c;1)/[2f'(b,c;1) + f''(b,c;1)]. \tag{2.2}$$

For fixed $m = f'(b,c;1)$, c is an increasing function of $f''(b,c;1)$, and if s is also fixed $(0 \leqq s < 1)$,

$$\frac{d}{dc}f(b,c;s) = \frac{d}{dc}f((1-c)^2m,c;s) = \frac{m(1-s)^2}{(1-cs)^2} > 0,$$

so that (i) holds. Similarly, since $r_0 = 1 - b/(1-c)$, we see that

$$b = (1-r_0)^2/f'(b,c;1)$$

and

$$c = (f'(b,c;1) + r_0 - 1)/f'(b,c;1).$$

Thus, for fixed $m = f'(b,c;1)$, $f(b,c;s)$ is an increasing function of r_0 for all s, $0 \leqq s < 1$, and (ii) holds. Notice that whenever $f'(b,c;1) \leqq 1$, $b + c \leqq 1$ for any $f''(b,c;1) > 0$ and $b + c \leqq 1$ for any proper $r_0 > 0$, so that the above arguments provide legitimate parameter values for the f.l.g.f. and also show that the class of f.l.g.f.'s provide a wide variety when $m \leqq 1$ for the bounding approach we use.

3. Subcritical and critical cases

Let g be any p.g.f. satisfying $m \leqq 1$. In this section, we show that the best bounding f.l.g.f.'s with mean m can be determined easily when g is in the class $A = \{p_0 + p_1 s + p_2 s^2\}$ or $B = \{p_0 + (1-p_0)s^k,\ k \geqq 1$ a real number$\}$. We then use these two classes to obtain f.l.g.f. bounds for a p.g.f. g of more arbitrary form.

Lemma 2. (i) $f(b_1, c_1; s)$ with $b_1 = (p_1 + 2p_2)^3(p_1 + 3p_2)^{-2}$ and $c_1 = p_2/(p_1 + 3p_2)$ is the best lower bounding f.l.g.f. for $p_0 + p_1 s + p_2 s^2$ with mean $m = p_1 + 2p_2$.

(ii) $f(b_1', c_1'; s)$ with $b_1' = (p_1 + p_2)^2/(p_1 + 2p_2)$ and $c_1' = p_2/(p_1 + 2p_2)$ is the best upper bounding f.l.g.f. for $p_0 + p_1 s + p_2 s^2$ with mean $m = p_1 + 2p_2$.

Proof. (i) $f(b, c; s) \leqq p_0 + p_1 s + p_2 s^2$, $0 \leqq s \leqq 1$, and $b(1-c)^{-2} = p_1 + 2p_2$ if and only if

$$1 - (1-c)(p_1 + 2p_2) + (1-c)^2(p_1 + 2p_2)s/(1 - cs)$$
$$\leqq p_0 + p_1 s + p_2 s^2, \; 0 \leqq s \leqq 1.$$

This is true if and only if $c \leqq p_2/(p_1 + 2p_2 + p_2 s)$, $0 \leqq s \leqq 1$. That is, $f(b, c; s) \leqq p_0 + p_1 s + p_2 s^2$, $0 \leqq s \leqq 1$, and $f'(b, c; 1) = p_1 + 2p_2$ if and only if $0 \leqq c \leqq p_2/(p_1 + 3p_2)$ and $b = (1-c)^2(p_1 + 2p_2)$. Thus, $f(b_1, c_1; s) \leqq p_0 + p_1 s + p_2 s^2$, $0 \leqq s \leqq 1$, and since $f'(b_1, c_1; 1) = p_1 + 2p_2$ and $f''(b_1, c_1; 1) = 2p_2$, $f(b_1, c_1; s)$ is the best lower bounding f.l.g.f. with mean $p_1 + 2p_2$, by Lemma 1 (i).

(ii) Similarly, $f(b, c; s) \geqq p_0 + p_1 s + p_2 s^2$, $0 \leqq s \leqq 1$, and $b(1-c)^{-2} = p_1 + 2p_2$ if and only if $p_2/(p_1 + 2p_2) \leqq c < 1$ and $b = (1-c)^2(p_1 + 2p_2)$. Thus, $f(b_1', c_1'; s) \geqq p_0 + p_1 s + p_2 s^2$, $0 \leqq s \leqq 1$, and since $f'(b_1', c_1'; 1) = p_1 + 2p_2$ and $f(b_1', c_1'; 0) = p_0$, $f(b_1', c_1'; s)$ is the best upper bounding f.l.g.f. with mean $p_1 + 2p_2$, by Lemma 1 (ii).

Lemma 3. (i) $f(b_2, c_2; s)$ with $b_2 = (1 - p_0)/k$ and $c_2 = (k-1)/k$ is the best upper bounding f.l.g.f. for $p_0 + (1 - p_0)s^k$ with mean $m = k(1 - p_0)$.

(ii) $f(b_2', c_2'; s)$ with $b_2' = 4k(1 - p_0)(k+1)^{-2}$ and $c_2' = (k-1)/(k+1)$ is the best lower bounding f.l.g.f. for $p_0 + (1 - p_0)s^k$ with mean $m = k(1 - p_0)$.

Proof. (i) $p_0 + (1 - p_0)s^k \leqq f(b_2, c_2; s)$, $0 \leqq s \leqq 1$, if and only if

$$p_0 + (1 - p_0)s^k \leqq p_0 + (1 - p_0)s/(k - (k-1)s), \quad 0 \leqq s \leqq 1,$$

which holds if and only if $t_1(s) = 1 - s^k - (1 - s)ks^{k-1} \geqq 0$, $0 \leqq s \leqq 1$. Now $t_1(1) = 0$ and $t_1'(s) = -k(k-1)s^{k-2}(1-s) \leqq 0$, $0 \leqq s \leqq 1$, so that $t_1(s) \geqq 0$, $0 \leqq s \leqq 1$. Also, since $f'(b_2, c_2; 1) = k(1 - p_0)$ and $f(b_2, c_2; 0) = p_0$, $f(b_2, c_2; s)$ is the best upper bounding f.l.g.f. for $p_0 + (1 - p_0)s^k$ with mean $k(1 - p_0)$, by Lemma 1 (ii).

(ii) $f(b_2', c_2'; s) \leqq p_0 + (1 - p_0)s^k$, $0 \leqq s \leqq 1$, if and only if

$$p_0 + (1 - p_0)s^k \geqq 1 - [2k(1 - p_0)/(k+1)]$$
$$+ [4k(1 - p_0)s/((k+1)^2 - (k^2 - 1)s)], \; 0 \leqq s \leqq 1,$$

which holds if and only if $t_2(s) = (k-1)(1 - s^{k+1}) - (k+1)s(1 - s^{k-1}) \geqq 0$, $0 \leqq s \leqq 1$. Now $t_2(1) = 0$, $t_2'(1) = 0$, and $t_2''(s) \geqq 0$, so that $t_2(s) \geqq 0$, $0 \leqq s \leqq 1$. Also, since $f'(b_2', c_2'; 1) = k(1 - p_0)$ and $f''(b_2', c_2'; 1) = k(k-1)(1 - p_0)$, $f(b_2', c_2'; s)$ is the best lower bounding f.l.g.f. for $p_0 + (1 - p_0)s^k$ with mean $k(1 - p_0)$, by Lemma 1(i).

The f.l.g.f.'s $f(b_1, c_1; s)$ and $f(b_1', c_1'; s)$ or $f(b_2, c_2; s)$ and $f(b_2', c_2'; s)$ can be used as described in Section 2 to provide bounds for the extinction time distribution of the subcritical or critical Galton-Watson process with p.g.f. $p_0 + p_1 s + p_2 s^2$ or $p_0 + (1 - p_0)s^k$, respectively. The simple approach used in Lemmas 2 and 3 becomes very cumbersome as the number of terms in the p.g.f. increases. Instead, we now direct our attention to finding f.l.g.f.'s that bound a more arbitrary form of p.g.f. In Lemma 4 we show that a p.g.f. g with mean $m \leqq 1$ can be bounded below by a p.g.f. from class A with mean m, and bounded above by a function from class B with first derivative at one equal to m, if $g''(1) < \infty$. The bounding functions we derive are the best in these classes which bound g and have derivative m at $s = 1$.

Lemma 4. Let $g(s) = \sum_{j=0}^{\infty} p_j s^j$, with $m = g'(1) \leqq 1$.

(i) If $p_0 \leqq 1 - \frac{1}{2}m$, then

$$g(s) \geqq L(s) \stackrel{\text{def}}{=} p_0 + [2(1 - p_0) - m]s + (p_0 + m - 1)s^2, \; 0 \leqq s \leqq 1.$$

If $p_0 \geqq 1 - \frac{1}{2}m$, then

$$g(s) \geqq L(s) \stackrel{\text{def}}{=} 1 - \tfrac{1}{2}m + \tfrac{1}{2}ms^2, \; 0 \leqq s \leqq 1.$$

$L(s)$ is the best lower bound in class A with mean m.

(ii) If $g''(1) < \infty$, then

$$g(s) \leqq U(s) \stackrel{\text{def}}{=} 1 - \frac{m^2}{m + g''(1)} + \frac{m^2}{m + g''(1)} s^{(g''(1)/m)+1}, \; 0 \leqq s \leqq 1.$$

$U(s)$ is the best upper bound in class B with first derivative at one equal to m.

Proof. (i) Suppose that $p_0 \leqq 1 - \frac{1}{2}m$. Then

$$L(s) = p_0 + [2(1 - p_0) - m]s + (p_0 + m - 1)s^2 \leqq g(s), \quad 0 \leqq s \leqq 1,$$

if and only if $\sum_{j=3}^{\infty} p_j[j - 2 - (j-1)s + s^{j-1}] \geqq 0$, $0 \leqq s \leqq 1$. Let $v_k(s) = k - 2 - (k-1)s + s^{k-1}$, $k \geqq 3$. Then $v_k(0) = k - 2 > 0$, $v_k(1) = 0$, and $v_k'(s) = -(k-1)(1 - s^{k-2}) \leqq 0$, $0 \leqq s \leqq 1$, $k \geqq 3$. Hence $v_k(s) \geqq 0$, $0 \leqq s \leqq 1$ and $k \geqq 3$, so

$$\sum_{j=3}^{\infty} p_j[j - 2 - (j-1)s + s^{j-1}] \geqq 0, \; 0 \leqq s \leqq 1,$$

and $L(s) \leqq g(s)$. Now suppose that $p_0 \geqq 1 - \frac{1}{2}m$. Then $L(s) = 1 - \frac{1}{2}m + \frac{1}{2}ms^2 \leqq g(s)$, $0 \leqq s \leqq 1$, if and only if

$$\tfrac{1}{2}m(1 - s^2) - \sum_{j=1}^{\infty} p_j(1 - s^j) \geqq 0,$$

which holds if and only if

$$v(s) = \tfrac{1}{2}m(1+s) - [p_1 + p_2(1+s) + p_3(1+s+s^2) + \cdots + p_j(1+s+s^2+\cdots+s^{j-1}) + \cdots] \geqq 0.$$

Now $v(0) = \frac{1}{2}m - (1-p_0) \geqq 0$, $v(1) = 0$ and $v''(s) \leqq 0$, $0 \leqq s \leqq 1$. Since v is concave, $v(s) \geqq 0$, $0 \leqq s \leqq 1$, and hence $L(s) \leqq g(s)$, $0 \leqq s \leqq 1$.

To show that L is the best lower bound in class A with mean m, fix $m > 0$ and let $A = \{q_0 + q_1 s + q_2 s^2\}$. In terms of $m = q_1 + 2q_2$ and q_0, we can express $q_1 = 2(1-q_0) - m$ and $q_2 = m + q_0 - 1$, so that

$$q_0 + q_1 s + q_2 s^2 = q_0 + [2(1-q_0) - m]s + [m + q_0 - 1]s^2, \; 0 \leqq s \leqq 1.$$

Now

$$q_0' + [2(1-q_0') - m]s + [m + q_0' - 1]s^2 \geqq q_0 + [2(1-q_0) - m]s + [m + q_0 - 1]s^2, \qquad 0 \leqq s \leqq 1,$$

if and only if $(q_0' - q_0)(1-s)^2 \geqq 0$, $0 \leqq s \leqq 1$, or $q_0' \geqq q_0$. That is, for fixed m, the best lower bounding p.g.f. in class A with mean m is the one with largest q_0, or equivalently since $q_2 = m + q_0 - 1$, the largest q_2. If $p_0 \geqq 1 - \frac{1}{2}m$, then $q_0 = 1 - \frac{1}{2}m$ and $q_2 = \frac{1}{2}m$ are clearly the largest values q_0 and q_2 may assume under the restriction $q_1 + 2q_2 = m$. If $p_0 \leqq 1 - \frac{1}{2}m$, the largest values of q_0 and q_2 possible so that $q_0 \leqq p_0$ and $q_1 + 2q_2 = m$ are clearly $q_0 = p_0$ and $q_2 = m + q_0 - 1$. In each case this is $L(s)$, which has been shown to be a lower bound for g, and hence is the best in class A with mean m.

(ii) Assume now that $g''(1) < \infty$. Then

$$U(s) = 1 - \frac{m^2}{m + g''(1)} + \frac{m^2}{m + g''(1)} s^{(g''(1)/m)+1} \geqq g(s), \qquad 0 \leqq s \leqq 1,$$

if and only if

$$w(s) = m(1 - s^{(g''(1)/m)+1}) - ((g''(1)/m) + 1)(1 - g(s)) \leqq 0, \quad 0 \leqq s \leqq 1.$$

Now $w(1) = 0$, so a sufficient condition that $w(s) \leqq 0$, $0 \leqq s \leqq 1$, is that

$$w'(s) = ((g''(1)/m) + 1)(g'(s) - ms^{g''(1)/m}) \geqq 0, \qquad 0 \leqq s \leqq 1.$$

That is, it is sufficient to show that $g'(s)/m \geqq s^{g''(1)/m}$, $0 \leqq s \leqq 1$, or

$$\sum_{j=1}^{\infty} \frac{jp_j}{m} s^{j-1} \geqq s^{\sum_{j=2}^{\infty}(j-1)jp_j/m}.$$

Now consider the random variable Y such that $P(Y = j-1) = jp_j/m, j \geqq 1$. Then

$$\sum_{j=1}^{\infty} \frac{jp_j}{m} s^{j-1} = Es^Y \geqq s^{EY} = s^{\sum_{j=2}^{\infty}(j-1)jp_j/m}$$

by Jensen's inequality, for $0 \leqq s \leqq 1$. Thus $U(s) \geqq g(s)$, $0 \leqq s \leqq 1$.

For fixed $m > 0$ and fixed s, $1 - m(n+1)^{-1} + m(n+1)^{-1}s^{n+1}$ is an increasing function of n. Thus $1 - m(n+1)^{-1} + m(n+1)^{-1}s^{n+1} \geqq g(s)$ whenever

$n \geqq g''(1)/m$, and out of these values the best bound is for $n = g''(1)/m$. Also, if $n < g''(1)/m$, then the second derivative at one is less than $g''(1)$, so $1-m(n+1)^{-1} + m(n+1)^{-1}s^{n+1}$ is not an upper bound for g. Hence $U(s)$ is the best upper bound in class B with first derivative at one equal to m.

Thus, any p.g.f. g with $m \leqq 1$ and $g''(1) < \infty$ can be easily bounded by two functions with first derivative at one equal to m, which are themselves amenable to bounding by f.l.g.f.'s with mean m. This fact enables us to display f.l.g.f. bounds with mean m for this very general form of p.g.f.

Lemma 5. Let $g(s) = \sum_{j=0}^{\infty} p_j s^j$ be a p.g.f. with mean m.

(i) If $m \leqq 1$, $g(s) \geqq f(b_1, c_1; s)$, $0 \leqq s \leqq 1$, where $b_1 = m^3(p_0 + 2m - 1)^{-2}$ and $c_1 = (p_0 + m - 1)/(p_0 + 2m - 1)$, $p_0 \leqq 1 - \frac{1}{2}m$; $b_1 = 4m/9$ and $c_1 = \frac{1}{3}$, $p_0 \geqq 1 - \frac{1}{2}m$.

(ii) If $g''(1) < \infty$, $g(s) \leqq f(b_2, c_2; s)$, $0 \leqq s \leqq 1$, where $b_2 = m^3(g''(1) + m)^{-2}$, $c_2 = g''(1)/(g''(1) + m)$.

Proof. (i) If $p_0 \leqq 1 - \frac{1}{2}m$,

$$g(s) \geqq L(s) = p_0 + [2(1 - p_0) - m]s + (p_0 + m - 1)s^2, \qquad 0 \leqq s \leqq 1,$$

by Lemma 4(i). Also, $L(s) \geqq f(b_1, c_1; s)$, $0 \leqq s \leqq 1$, where $b_1 = m^3(p_0 + 2m - 1)^{-2}$ and $c_1 = (p_0 + m - 1)/(p_0 + 2m - 1)$, by Lemma 2(i). Hence, $g(s) \geqq f(b_1, c_1; s)$, where $g'(1) = f'(b_1, c_1; 1) = m$ and $s_0(b_1, c_1) = 1 + m(1 - m)/(p_0 + m - 1)$.

If $p_0 \geqq 1 - \frac{1}{2}m$, $g(s) \geqq L(s) = 1 - \frac{1}{2}m + \frac{1}{2}ms^2$, $0 \leqq s \leqq 1$, by Lemma 4(i). Again, by Lemma 2(i), $L(s) \geqq f(b_1, c_1; s)$, $0 \leqq s \leqq 1$, where $b_1 = 4m/9$ and $c_1 = \frac{1}{3}$. Hence $g(s) \geqq f(b_1, c_1; s)$, where $g'(1) = f'(b_1, c_1; 1) = m$ and $s_0(b_1, c_1) = 1 + 2(1 - m)$. Thus, when $m \leqq 1$, $g(s) \geqq f(b_1, c_1; s)$, where $f'(b_1, c_1; 1) = m$ and $s_0(b_1, c_1) = 1 + D(1 - m)$, where $D = \max[2, m/(p_0 + m - 1)]$.

(ii) If $g''(1) < \infty$,

$$g(s) \leqq U(s) = 1 - \frac{m^2}{m + g''(1)} + \frac{m^2}{m + g''(1)} s^{(g''(1)/m)+1}, \; 0 \leqq s \leqq 1,$$

by Lemma 4(ii). Also, $U(s) \leqq f(b_2, c_2; s)$, $0 \leqq s \leqq 1$, where $b_2 = m^3(g''(1) + m)^{-2}$ and $c_2 = g''(1)/(m + g''(1))$, by Lemma 3(ii). Hence, $g(s) \leqq f(b_2, c_2; s)$, $0 \leqq s \leqq 1$, where $g'(1) = f'(b_2, c_2; 1) = m$ and $s_0(b_2, c_2) = 1 + m(1 - m)/g''(1)$.

As a result of Lemma 5, relatively simple bounds can be formed for $P(T \leqq n)$, ξ_p, ET and μ that depend only upon m, p_0 and $g''(1)$. The form of the bounds follows directly from the extinction time distributions associated with the bounding f.l.g.f.'s.

If $m \neq 1$, the nth iterate of Expression (2.1) for $f(b, c; s)$ is

$$f_n(b, c; s) = 1 - \frac{m^n(s_0 - 1)}{s_0 - m^n} + \frac{m^n((s_0 - 1)/(s_0 - m^n))^2 s}{1 - ((1-m^n)/(s_0 - m^n))s};$$

if $m = 1$,

$$f_n(b,c;s) = \frac{nc - [(n+1)c - 1]s}{1 + (n-1)c - ncs}.$$

Thus, letting $T(b,c)$ denote the extinction time of a Galton-Watson process with p.g.f. $f(b,c;s)$,

(i) if $m \neq 1$,

$$P(T(b,c) \leqq n) = s_0(1 - m^n)/(s_0 - m^n), \qquad n \geqq 1, \tag{3.1}$$

(ii) if $m = 1$,

$$P(T(b,c) \leqq n) = nc/(1 + (n-1)c), \qquad n \geqq 1. \tag{3.2}$$

We can also calculate explicitly the percentiles of the distribution of $T(b,c)$. Considering $P(T(b,c) \leqq n)$ as a continuous function of n for all real numbers $n \geqq 0$, the 100pth percentile, denoted by $\xi_p(b,c)$, is the solution of

$$p = s_0(1 - m^{\xi_p(b,c)})/(s_0 - m^{\xi_p(b,c)}) \ (m \neq 1),$$

$$p = \xi_p(b,c)c/(1 + (\xi_p(b,c) - 1)c) \ (m = 1).$$

That is,

$$\xi_p(b,c) = \log(s_0(1-p)/(s_0 - p))/\log m, \, 0 < p < q, \, m \neq 1, \tag{3.3}$$

$$\xi_p(b,c) = p(1-c)/(1-p)c, \ 0 < p < 1, \ m = 1. \tag{3.4}$$

These equations express the percentiles as a continuous function of p. If we require $\xi_p(b,c)$ to be the smallest integer such that $P(T(b,c) \leqq \xi_p(b,c)) \geqq p$, then $\xi_p(b,c)$ is the greatest integer part of (3.3) or (3.4) plus 1, for $p \neq f_n(b,c;0)$, $n \geqq 1$. With this in mind, however, we shall use the continuous interpretation for simplicity. Now, since

$$f(b_1,c_1;s) \leqq g(s) \leqq f(b_2,c_2;s), \qquad 0 \leqq s \leqq 1,$$

we have

$$P(T(b_1,c_1) \leqq \xi_p) \leqq P(T \leqq \xi_p) = p \leqq P(T(b_2,c_2) \leqq \xi_p).$$

Then, since

$$p = P(T(b_1,c_1) \leqq \xi_p(b_1,c_1)) = P(T(b_2,c_2) \leqq \xi_p(b_2,c_2)),$$

we have

$$\xi_p(b_2,c_2) \leqq \xi_p \leqq \xi_p(b_1,c_1), \qquad 0 < p < q. \tag{3.5}$$

That is, fractional linear bounds for g extend directly to bounds for the percentiles of T.

The bounds in Theorem 1 are now a consequence of (1.1)–(1.3) and (3.1)–(3.5), Lemma 5 and the expressions in its proof for $s_0(b_1,c_1)$ and $s_0(b_2,c_2)$.

Theorem 1. Let $g(s) = \sum_{j=0}^{\infty} p_j s^j$ be a p.g.f. with mean $m \leqq 1$ and with $g''(1) < \infty$, and let $D = \max[2, m/(p_0 + m - 1)]$. Then,

$$\frac{[1 + D(1-m)]\,(1-m^n)}{1 + D(1-m) - m^n} \leqq P(T \leqq n)$$

$$\leqq \frac{[1 + (m(1-m)/g''(1))]\,(1-m^n)}{1 + (m(1-m)/g''(1)) - m^n},\ n \geqq 1,\ m < 1;$$

$$\frac{n}{n+D} \leqq P(T \leqq n) \leqq \frac{n}{n + [g''(1)]^{-1}},\ n \geqq 1,\ m = 1;$$

$$\log\left[\frac{(1-p)\,[g''(1) + m(1-m)]}{g''(1)\,(1-p) + m(1-m)}\right] \Big/ \log m \leqq \xi_p$$

$$\leqq \log\left[\frac{(1-p)\,[1 + D(1-m)]}{1 - p + D(1-m)}\right] \Big/ \log m, \qquad m < 1,\ \ 0 < p < 1;$$

$$\frac{p}{(1-p)g''(1)} \leqq \xi_p \leqq \frac{pD}{1-p},\ m = 1,\ 0 < p < 1;$$

$$1 + \frac{m^2}{2(g''(1)+m)} + m(1-m)\log\left[1 - \frac{mg''(1)}{g''(1) + m(1-m)}\right] \Big/ g''(1)\log m$$

$$\leqq ET \leqq 1 + \frac{DM}{D+1} + D(1-m)\log\left[1 - \frac{m^{\frac{3}{2}}}{1 + D(1-m)}\right] \Big/ \log m;$$

$$1 + \frac{1}{D(1-m)} \leqq \mu \leqq 1 + \frac{g''(1)}{m(1-m)}.$$

If $g''(1) = \infty$, the side of each of these bounds that corresponds to bounding g below by $f(b_1, c_1; s)$ still applies. In fact, when $g''(1) = \infty$, it is possible to derive an upper bounding f.l.g.f. with mean slightly less than m, so that a complete set of bounds can be given even in that situation.

4. Supercritical case

When $m > 1$, $g_n(q) = q$ for $n \geqq 1$, $\lim_{n\to\infty} g_n(s) = q$ for $0 \leqq s < 1$, and

$$P(T > n) = q - g_n(0) \sim \mu^{-1}[g'(q)]^n \text{ as } n \to \infty.$$

A direct calculation shows that there is always exactly one f.l.g.f. that equals q and has derivative $g'(q)$ at $s = q$. It is unrealistic, then, to expect two f.l.g.f.'s that bound g over $0 \leqq s \leqq q$ to produce good bounds for the extinction time distribution. However, it is possible to derive bounds with quality comparable to those derived in Section 3 by exploiting the following duality between subcritical and supercritical processes. Let

$$\bar{g}(s) = g(qs)/q.$$

Then $\bar{g}'(1) = g'(q) < 1$, so $\bar{g}$ is the p.g.f. of a subcritical process, and can be given the interpretation of being the p.g.f. of the original process conditioned on eventual extinction (see Athreya and Ney (1973)).

Now

$$E(s^{Z_n} \mid T < \infty) = \bar{g}_n(s) = g_n(qs)/q,$$

so that

$$g_n(0) = q\bar{g}_n(0).$$

Thus, bounds for the extinction time distribution can be obtained by using the f.l.g.f. bounds derived in the last section for a p.g.f. with mean $g'(q) < 1$. Also, $\bar{g}''(1) = qg''(q) < \infty$, so that $f(b_2, c_2; s)$ always applies as an upper bound for $\bar{g}$. Therefore, letting

$$b_1 = g'(q)^3 [p_0 q^{-1} + 2g'(q) - 1]^{-2}$$

and

$$c_1 = (p_0 q^{-1} + g'(q) - 1)/(p_0 q^{-1} + 2g'(q) - 1)$$

when $p_0 q^{-1} \leqq 1 - \frac{1}{2}g'(q)$, letting $b_1 = 4g'(q)/9$ and $c_1 = \frac{1}{3}$ when $p_0 q^{-1} \geqq 1 - \frac{1}{2}g'(q)$, and letting $b_2 = g'(q)^3[qg''(q) + g'(q)]^{-2}$ and $c_2 = qg''(q)/(qg''(q) + g'(q))$, we have from Lemma 5 that

$$qf_n(b_1, c_1; 0) \leqq P(T \leqq n) \leqq qf_n(b_2, c_2; 0)$$

and

$$ET(b_2, c_2) \leqq E[T \mid T < \infty] \leqq ET(b_1, c_1).$$

Also, $P(T \leqq n) = p$ $(0 < p < q)$ if and only if $P(T \leqq n \mid T < \infty) = pq^{-1}$, so that

$$\xi_{p/q}(b_2, c_2) \leqq \xi_p \leqq \xi_{p/q}(b_1, c_1).$$

5. Fractional linear bounds for the Poisson p.g.f.

Naturally, in some instances there are f.l.g.f.'s which provide a tighter bound for g than those in Lemma 5, thus giving better bounds for the extinction time distribution. In fact, for many p.g.f.'s, it is possible to derive the best bounding f.l.g.f.'s with mean m. We have seen that such is the situation for p.g.f.'s of the form $p_0 + p_1 s + p_2 s^2$ or $p_0 + (1 - p_0)s^k$, $k \geqq 1$. As another example, we derive now the best f.l.g.f. bounds with mean λ for the Poisson p.g.f. $g(\lambda; s) = e^{\lambda(s-1)}$, $0 \leqq \lambda \leqq 1$.

Theorem 2. When $\lambda \leqq 1$, the best upper and lower bounding f.l.g.f.'s for $g(\lambda; s) = e^{\lambda(s-1)}$ with mean λ have the parametrizations $b = (1 - e^{-\lambda})^2/\lambda$, $c = (\lambda + e^{-\lambda} - 1)/\lambda$ and $b = \lambda(2/(\lambda + 2))^2$, $c = \lambda/(\lambda + 2)$, respectively.

Proof. Let $0 < \lambda \leqq 1$ be fixed. $f(b, c; s) \geqq e^{\lambda(s-1)}$, $0 \leqq s \leqq 1$, and $g'(\lambda; 1) = f'(b, c; 1)$ (i.e., $b = (1 - c)^2\lambda$) if and only if

$$1-(1-c)\,\lambda+(1-c)^2\lambda s/(1-cs)\geqq e^{\lambda(s-1)},\qquad 0\leqq s\leqq 1,$$

which is equivalent to

$$c\geqq\frac{e^{\lambda(s-1)}-1+\lambda(1-s)}{se^{\lambda(s-1)}-s+\lambda(1-s)}=v(\lambda,s),\qquad 0\leqq s<1.$$

Now $v(\lambda;0)=(e^{-\lambda}+\lambda-1)/\lambda$, $v(\lambda,1^-)=\lambda/(\lambda+2)$, and it can be easily seen that $v(\lambda;0)>v(\lambda;1^-)$, all $\lambda>0$. Also,

$$\frac{d}{ds}v(\lambda;s)=v'(\lambda;s)=\frac{[e^{\frac{1}{2}\lambda(s-1)}\lambda(1-s)]^2-[1-e^{\lambda(s-1)}]^2}{[se^{\lambda(s-1)}-(1+\lambda)s+\lambda]^2}.$$

so that $v'(\lambda;s)=0$ implies that $e^{\frac{1}{2}\lambda(s-1)}\lambda(1-s)=1-e^{\lambda(s-1)}$, or, equivalently, $\frac{1}{2}\lambda(1-s)=\frac{1}{2}[e^{\frac{1}{2}\lambda(1-s)}-e^{-\frac{1}{2}\lambda(1-s)}]$. But $x=\frac{1}{2}(e^x-e^{-x})$ (or $\sinh x=x$) if and only if $x=0$, so $v'(\lambda;s)=0$ implies that $s=1$. Thus, $v(\lambda;s)$ must be a strictly monotone function of s over $[0,1)$. Since $v(\lambda;0)>v(\lambda;1^-)$, we see that $v'(\lambda;s)<0$, $0\leqq s<1$, and

$$\sup_{0\leqq s<1}v(\lambda;s)=v(\lambda;0)=(e^{-\lambda}+\lambda-1)/\lambda,$$

$$\inf_{0\leqq s<1}v(\lambda;s)=v(\lambda;1^-)=\lambda/(\lambda+2).$$

As a consequence, $f(b,c;s)\geqq e^{\lambda(s-1)}$, $0\leqq s\leqq 1$, and $f'(b,c;1)=\lambda$ if and only if $(e^{-\lambda}+\lambda-1)/\lambda\leqq c<1$ and $b=(1-c)^2\lambda$. Note that when $b(1-c)^{-2}=\lambda$, $c<1$ and $b+c=(1-c)^2\lambda+c\leqq 1$ if and only if $(\lambda-1)/\lambda<c<1$. Since $(e^{-\lambda}+\lambda-1)/\lambda>0\geqq(\lambda-1)/\lambda$ for $0<\lambda\leqq 1$, the set of points $\{(e^{-\lambda}+\lambda-1)/\lambda\leqq c<1$ and $b=(1-c)^2\lambda\}$ give proper parameter values for the f.l.g.f. Now $c=(e^{-\lambda}+\lambda-1)/\lambda$ and $b=(1-c)^2\lambda=(1-e^{-\lambda})^2/\lambda$ are such that $f'(b,c;1)=\lambda$ and $f(b,c;0)=g(\lambda;0)=e^{-\lambda}$. Thus, this parametrization gives the best upper bounding f.l.g.f. for g with mean λ, by Lemma 1(ii).

Similarly, $f(b,c;s)\leqq e^{\lambda(s-1)}$, $0\leqq s\leqq 1$, and $g'(\lambda;1)=f'(b,c;1)$ if and only if $0\leqq c\leqq\inf_{0\leqq s<1}v(\lambda;s)=\lambda/(\lambda+2)$, $b=(1-c)^2\lambda$ and $b+c\leqq 1$, where $v(\lambda;s)$ is as above. Thus, when $0<\lambda\leqq 1$ and $b=\lambda(2/(\lambda+2))^2$ and $c=\lambda/(\lambda+2)$, $f(b,c;s)\leqq g(\lambda;s)$, $0\leqq s\leqq 1$. Then, since $f'(b,c;1)=\lambda$ and $f''(b,c;1)=\lambda^2=g''(\lambda;1)$, this parametrization gives the best lower bounding f.l.g.f. for g with mean λ, by Lemma 1 (i).

Seneta (1967) also derived the above lower bound for g. Bounds for $P(T\leqq n)$, ξ_p, ET and μ can be obtained by combining Theorem 2 with Expressions (1.1)–(1.3) and (3.1)–(3.5). Table 1 contains bounds for $P(T\leqq n)$ when $\lambda=1$, and Table 2 contains bounds for ET for several values of λ. The bounds for ET compare favorably to those given by Heathcote and Seneta (1966), Seneta (1967), and Pollak (1971). The supercritical process can be treated as in Section 4, noting that

$$\bar{g}(s) = q^{-1}g(\lambda; qs) = q^{-1}e^{\lambda(qs-1)}$$
$$= e^{\lambda q(s-1)}q^{-1}e^{\lambda(q-1)} = e^{\lambda q(s-1)} = g(\lambda q; s),$$

where $\lambda q = g'(\lambda; q) < 1$.

TABLE 1

$P(T \leqq n)$: Critical Poisson branching process

n	5	10	15	20	50	100	200	500
lower bound	0.714	0.833	0.882	0.909	0.961	0.9804	0.9901	0.9960
true value	0.732	0.842	0.888	0.913	0.963	0.9807	0.9902	0.9961
upper bound	0.744	0.853	0.897	0.921	0.967	0.9830	0.9915	0.9966

TABLE 2

ET: Poisson branching process

λ	0.50	0.70	0.90	0.95	0.99	0.999
lower bound	1.71	2.33	3.86	4.93	7.67	11.5
true value	1.74	2.37	4.00	5.16		
upper bound	1.76	2.38	4.10	5.25	8.43	13.0

6. Asymptotic forms of the bounds

In each of the three special cases we have discussed, the upper bound for $P(T > n)$ in the critical case is $2/(2 + g''(1)n)$, which is smaller than Kolmogorov's approximation of $P(T > n) \sim 2/(g''(1)n)$ for any $n \geqq 1$. In fact, whenever the f.l.g.f. $f(b^*, c^*; s)$ with $f'(b^*, c^*; 1) = 1$ and $f''(b^*, c^*; 1) = g''(1)$ bounds g, the corresponding bound for $P(T > n)$ is $2/(2 + g''(1)n)$ by (2.2) and (3.2). Also, for any $m \leqq 1$, it can be seen that one side of Pollak's ((1971), Inequality 3.2) bound for $q - g_n(s)$ applies if and only if the f.l.g.f. $f(b^*, c^*; s)$ with $f'(b^*, c^*; 1) = m$ and $f''(b^*, c^*; 1) = g''(1)$ bounds g, in which case his bound is identical to the one obtained using the bounding method described in this paper. Then, the resulting bound on the extinction time distribution has the asymptotic form as $m \to 1^-$ given by Seneta (1968), if g is a member of a class of p.g.f.'s satisfying conditions stated in that paper. Under similar conditions, Nagaev and Muhamedhanova (1968) derived the approximation

$$1 - g_n(s) \doteqdot \frac{2(1-m)m^n(1-s)}{2(1-m) + g''(1)(1-s)(1-m^n)} \quad \text{as } n \to \infty \text{ and } m \to 1^-.$$

This is similar to the expression for $1 - f_n(b^*, c^*; s)$, parametrized in terms of m and $g''(1)$ instead of b^* and c^*.

7. Generalizations and extensions

The main advantages of the bounds derived in this paper are that they apply to nearly all Galton-Watson processes, they are good in the region of greatest applicational interest ($m \approx 1$), and yet they are relatively simple. Moreover, the f.l.g.f. bounds for a p.g.f. derived in this paper are useful for obtaining information about the extinction time distribution of more complex discrete time branching processes. A future paper will be devoted to showing how they can be used to derive bounds for the non-homogeneous Galton-Watson process and a branching process with random environment. The results in this paper apply directly to the age-dependent branching process, in the sense that the sizes of the successive generations of that process form a Galton-Watson process (see Harris (1963), p. 127).

The approach of bounding a p.g.f. by f.l.g.f.'s is in itself conceptually appealing. Spitzer (see Athreya and Ney (1973)) bounded a p.g.f. tightly within a neighborhood of $s = 1$ by f.l.g.f.'s to prove an important lemma for the critical Galton-Watson process. A variation of this approach can be used to derive simple proofs for asymptotic results as $m \to 1^-$ such as those stated by Seneta (1968), and may be useful in other aspects of the theory of branching processes. Lastly, if the p.g.f. in some evolving branching process is unknown, we can utilize Stigler's results (1971) on the estimation of the p.g.f. g in order to derive asymptotically normal estimates (as the number of observed offspring counts increases) of the bounds for the extinction time distribution.

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