Chapter 20

Connections Between Loglinear Models and Generalized Rasch Models for Ordinal Responses

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1. Introduction

This article deals with modeling responses of subjects to a set of similar items that have the same ordinal scale. Table 1, taken from the 1989 General Social Survey conducted by the National Opinion Research Center in the U. S., illustrates the type of data. Subjects gave their opinions regarding government spending on (1) the environment, (2) health, (3) assistance to big cities, and (4) law enforcement, using the response scale (Too little, About right, Too much). Table 2 is a similar sort of table, taken from the same survey. Subjects gave their opinions on (1) early teens (age 14-16) having sex relations before marriage and (2) a man and a woman having sex relations before marriage, using the response scale (Always wrong, Almost always wrong, Wrong only sometimes, Not wrong at all).

Cities		1			2			3		
Law Enforce		1	2	3	1	2	3	1	2	3
Envir.	*Health*									
1	1	62	17	5	90	42	3	74	31	11
		(62.0)	(17.9)	(3.4)	(85.8)	(34.0)	(5.7)	(77.3)	(27.6)	(11.1)
	2	11	7	0	22	18	1	19	14	3
		(13.1)	(5.2)	(0.9)	(24.9)	(14.8)	(2.1)	(20.1)	(10.0)	(4.4)
	3	2	3	1	2	0	1	1	3	1
		(1.8)	(0.6)	(0.3)	(3.1)	(1.5)	(0.7)	(6.0)	(3.3)	(1.3)
2	1	11	3	0	21	13	2	20	8	3
		(12.3)	(4.9)	(0.8)	(23.5)	(13.9)	(2.0)	(19.0)	(9.4)	(4.2)
	2	1	4	0	6	9	0	6	5	2
		(3.6)	(2.1)	(0.3)	(10.2)	(9.0)	(1.0)	(6.9)	(4.7)	(2.8)
	3	1	0	1	2	1	1	4	3	1
		(0.4)	(0.2)	(0.1)	(1.0)	(0.7)	(0.4)	(2.2)	(2.1)	(1.1)
3	1	3	0	0	2	1	0	9	2	1
		(1.6)	(0.6)	(0.2)	(2.7)	(1.4)	(0.6	(5.3)	(2.9)	(1.1)
	2	1	0	0	2	1	0	4	2	0
		(0.4)	(0.2)	(0.1)	(1.0)	(0.7)	(0.4)	(2.1)	(2.0)	(1.0)
	3	1	0	0	0	0	0	1	2	3
		(0.1)	(0.1)	(0.0)	(0.3)	(0.3)	(0.2)	(0.6)	(0.7)	(3.0)

Table 1: Opinions* about government spending, with fitted values for ordinal quasi-symmetry model in parentheses.

* Data from 1989 General Social Survey, with categories 1 = too little, 2 = about right, 3 = too much.

We discuss models that enable one to compare the item response distributions. In Table 1, for instance, one might study whether subjects regarded spending as relatively higher on one item than the others. For such models, we show that parameters that describe item effects relate to main effect parameters in certain loglinear models for the joint distribution.

We begin by reviewing a connection between item parameters in the Rasch model and main effect parameters in a loglinear model satisfying the property of quasi symmetry. We then focus on generalizations to ordinal responses. For a model using adjacent-category logits, a related ordinal quasi-symmetry model has main effect terms that reflect the ordinality. An alternative generalization for ordinal responses uses cumulative logits. We also discuss simpler representations of the models when there are only two items. This leads to simple ways of expressing and testing marginal homogeneity for ordinal matched-pairs data in square contingency tables.

Teen	*Premarital Sex*			
Sex	1	2	3	4
1	141 (141)	34 (34.5)	72 (72.4)	109 (109.0)
2	4 (1.8)	5 (4.9)	23 (22.8)	38 (37.5)
3	1 (0.6)	0 (1.8)	9 (8.9)	23 (22.9)
4	0 (0.1)	0 (0.3)	1 (1.5)	15 (15)

Table 2: Opinions* about teenage sex and premarital sex, with fitted values in parentheses for cumulative logit model.

* Data from 1989 General Social Survey, with categories 1 = always wrong, 2 = almost always wrong 3 = wrong only sometimes, 4 = not wrong.

2. The Rasch Model and Quasi Symmetry

Suppose N subjects respond to k items that use the same $m+1$ categories, $0,1,\ldots,m$. For subject v and item i, let X_{vi} denote the response category. We make the usual assumption of local independence for the repeated responses by a subject. We first consider the binary-response case, $m=1$. The Rasch model for k binary items is

$$\text{logit}\left[p\left(X_{vi}=1\right)\right]=\theta_v+\sigma_i, \quad v=1,\ldots,N, \quad i=1,\ldots,k. \tag{1}$$

Cross-classifying responses on the k binary items yields a 2^k contingency table. The i^{th} dimension represents the two possible response outcomes for the i^{th} item. To ease notation, we take $k=4$. Let (a,b,c,d) denote a potential response pattern for the four items, where each possible outcome is 0 or 1. Let p_{abcd} denote the probability of this sequence for a randomly selected subject. Let n_{abcd} denote the number of subjects in the sample having response pattern (a,b,c,d), and let $m_{abcd}=Np_{abcd}$ denote its expected frequency. Loglinear models in this article treat $\{n_{abcd}\}$ as a multinomial sample of size N, with cell probabilities proportional to $\{m_{abcd}\}$.

When the Rasch model holds, Tjur (1982) showed that $\{m_{abcd}\}$ satisfy the loglinear model

$$\log m_{abcd}=\sigma_1 a+\sigma_2 b+\sigma_3 c+\sigma_4 d+\lambda_{abcd}, \tag{2}$$

cf. equation (7) in chapter 1.

The interaction term λ_{abcd} is identical for all permutations of its argument; for instance, $\lambda_{0001} = \lambda_{0010} = \lambda_{0100}\ \lambda_{1000}$. In this binary response case, this corresponds to having a separate parameter for each distinct sum of indices.

The parameters $\{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$ in (2) are identical to $\{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$ in the Rasch model (1). Model form (2) is the special case of the general loglinear model for a $2 \times 2 \times 2 \times 2$ table in which the main effect terms are distinct but the higher-order terms are symmetric in their indices. Because model (2) exhibits symmetry in its interaction term but not in the main effects, it is called a *quasi-symmetry* model. Fienberg (1981), Hatzinger (1989), and Kelderman (1984) discussed related connections between the Rasch model and loglinear models.

There are two common ways of using maximum likelihood (ML) to estimate item parameters in the Rasch model. *Conditional ML* estimators of $\{\sigma_i\}$ eliminate the subject parameters by a conditional probability argument in which statistics providing information about $\{\theta_v\}$ are kept fixed. *Marginal ML* estimation refers to an adaptation of the model that treats subject effects as random effects, rather than fixed effects. One assumes a particular parametric form for the distribution of the subject effects, such as normal with mean 0 and unknown standard deviation. One eliminates the subject effects by averaging with respect to this distribution, and then one estimates the item parameters using the marginal distribution. A nonparametric version of marginal ML assumes an unknown continuous distribution for the subject effects.

Tjur (1982) showed that the ordinary ML estimators of $\{\sigma_i\}$ in the quasi-symmetry model (2) are identical to the conditional ML estimators of $\{\sigma_i\}$ in the Rasch model (1). Tjur (1982) also proved that the ML estimators of $\{\sigma_i\}$ in the quasi-symmetry model are identical to those obtained in a slightly extended version of nonparametric marginal ML. Thus, ML estimates from the quasi-symmetry model are also conditional ML and extended nonparametric marginal ML estimates for the Rasch model.

Tjur's work referred to an extended form of nonparametric marginal ML estimates, but later papers showed strong connections between the actual nonparametric marginal ML estimates and conditional ML estimates for the Rasch model. Under the assumption that the Rasch model holds, de Leeuw and Verhelst (1986) showed that the probability that nonparametric marginal ML estimators are identical to conditional ML estimators (and hence also to quasi-symmetric loglinear ML estimators) converges to 1 as N increases, for a fixed number of items. Lindsay et al. (1991) strengthened this, showing the same result if the subject-effect distribution has at least $(k+1)/2$ support points.

The Rasch model has been generalized from binomial to multinomial responses (Andersen 1973, Rasch 1961). In the case of unordered response categories, it has form

$$\log\left[p\left(X_{vi} = x\right)/p\left(X_{vi} = m\right)\right] = \theta_{vx} + \sigma_{ix}, \quad x = 0,1,\ldots,m\ . \tag{3}$$

Similar connections with loglinear models occur for this model. The conditional ML estimates of the item effects are identical to estimates of main effect parameters in the general quasi-symmetry loglinear model for a $(m+1)^k$ contingency table (Conaway 1989). That loglinear model has form

$$\log m_{ab\ldots t} = \sigma_{1a} + \sigma_{2b} + \ldots + \sigma_{kt} + \lambda_{ab\ldots t}\ , \tag{4}$$

where the interaction term is symmetric in its indices (cf. equation (16) in chapter 1).

The *mutual independence* model is the special case of (4) without the interaction parameters; that is, it is a loglinear model with k sets of main effect parameters alone. The *complete symmetry* model is the special case of (4) in which the main effect terms are identical; that is, $p_{ab\ldots t}$ is identical for any permutation of (a,b,...,t). The quasi-symmetry model is a generalization of complete symmetry that permits different main effect parameters for each item, and hence marginal heterogeneity. When the quasi-symmetry model holds, complete symmetry is equivalent to marginal homogeneity (Caussinus, 1966; Darroch, 1981). The standard test of marginal homogeneity is based on comparing the fits of the quasi-symmetry and complete symmetry models, with $df = m(k-1)$.

3. An Ordinal Model Using Adjacent-Category Logits

For the ordinal-response case, we again use notation for $k = 4$ items, corresponding to Table 1. We first consider an ordinal model that has the adjacent-categories logit representation

$$\log\left[p\left(X_{vi} = x+1\right)/\,p\left(X_{vi} = x\right)\right] = \theta_{vx} + \sigma_i \;. \tag{5}$$

This is a special case of the nominal-scale model in which the item effects have the ordinal structure $\sigma_{ix+1} - \sigma_{ix} = \sigma_i$ for all x; that is, $\{\sigma_{ix}\}$ are linear in x. The item effects are assumed to be identical for each pair of adjacent categories. Complete symmetry is the special case of (5) with equal item parameters. A somewhat simpler model decomposes θ_{vx} in (5) into $\delta_x + \theta_v$ (Andersen 1973; Andrich 1978; Duncan 1984; Hout et al. 1987; Agresti 1993a, cf. equation(17) in chapter 1).

Generalizing Tjur (1982), Agresti (1993a) noted that conditional ML estimates and extended nonparametric marginal ML estimates of the item effects in model (5) are identical to the ordinary ML estimates obtained in fitting the loglinear model

$$\log m_{abcd} = a\sigma_1 + b\sigma_2 + c\sigma_3 + d\sigma_4 + \lambda_{abcd}\,, \tag{6}$$

where λ is permutationally invariant. This is a special case of the quasi-symmetry model that has linear structure for the main effects. It treats the main effects as variates, with equally-spaced scores, rather than qualitative factors. Each main effect term has a single parameter, rather than the m parameters in the more general model. We call model (6) the *ordinal quasi-symmetry* model, since it reflects the ordering of the response categories. Agresti (1993a) also showed that estimates of $\{\sigma_i\}$ for the model with simpler structure for θ_{vx} equal those for a simpler loglinear model in which the interaction parameter depends only on the sum of the scores for the k items.

The complete symmetry model is the special case of (6) in which $\sigma_1 = \ldots = \sigma_k$. When model (6) fits well, one can test marginal homogeneity using a likelihood-ratio test with $df = k-1$, based on comparing its fit to that of complete symmetry. The ML estimates of $\{\sigma_i\}$ in (6) have the same order as the sample mean responses (for equally-spaced scores) in the k one-way margins of the $(m+1)^k$ table.

To illustrate the ordinal logit model (5) and the associated ordinal quasi-symmetry model (6), we analyze Table 1. Table 3 shows the goodness of fit of several loglinear models. The ordinal quasi-symmetry model fits relatively well, with likelihood-ratio goodness-of-fit statistic equal to $G^2 = 64.90$, with $df = 63$. Table 1 displays fitted values for this model. There is a dramatic improvement compared to the complete symmetry model (which has $G^2 = 638.24$, $df = 66$), at the expense of only adding three independent parameters.

Model	*Likelihood-ratio statistic*	*Pearson statistic*	*Degrees of freedom*
Mutual independence	124.3	277.6	72
Complete symmetry	638.2	711.6	66
Ordinal quasi symmetry	64.9	70.5	63
Quasi symmetry	58.0	61.7	60

Table 3: Goodness of fit of loglinear models for Table 1

Denote item effects in the ordinal quasi-symmetry model by σ_C, σ_L, σ_H, σ_E. The estimated effects and asymptotic standard errors (ase), using the constraint $\hat{\sigma}_E = 0$, are $\hat{\sigma}_C = 1.941$ (ase = 0.118), $\hat{\sigma}_L = 0.372$ (ase = 0.104), $\hat{\sigma}_H = 0.059$ (ase = 0.108). One can interpret these as estimates of the corresponding item parameters in generalization (5) of the Rasch model. Aid for cities received substantially less support than aid for the other items. For instance, for each subject, the estimated odds that the response is „too much" rather than „about right," or „about right" rather than „too little," are $\exp(1.941) = 7.0$ times as high for cities as for the environment. All asymptotic standard errors of differences of estimates are about 0.11. To compare all 6 pairs of item parameters while maintaining a bound of 0.05 on the overall error probability, we used $0.05/6 = 0.0083$ for the α-level for each comparison. This analysis indicates significant differences between all pairs except σ_H and σ_E.

One can use software for loglinear models to fit ordinary and ordinal quasi-symmetry models. For instance, it is simple to fit the models using software for generalized linear models, such as GLIM or SAS (PROC GENMOD). See Agresti (1993a, 1993b, 1995) for examples of the use of GLIM and Agresti (1996, p. 277) for the use of SAS.

4. An Ordinal Model Using Cumulative Logits

An alternative model form for ordinal responses uses cumulative logits. For subject v and item i, the cumulative logit analog of model (5) has form

$$\log\left[p(X_{vi} \le x)/\left(1 - p(X_{vi} \le x)\right)\right] = \theta_{vx} + \sigma_i \,, \qquad (7)$$

$x = 0,1,\ldots,m$, $v = 1,\ldots,N$, $i = 1,\ldots,k$. For each subject, the odds that the response for item a falls below any fixed level are $(\sigma_a - \sigma_b)$ times the odds for item b. As in the adjacent-categories logit model, primary interest is in $\{\sigma_i\}$ rather than $\{\theta_{vx}\}$,which satisfy $\theta_{vx-1} \le \theta_{vx}$ for all x and v. This model has the proportional odds property, for which the k item effects $\{\sigma_i\}$ are identical at each x. For justification of a model having this property, see Anderson and Philips (1981). McCullagh (1977) discussed a related model for $k = 2$. Complete symmetry is the special case of this model with equal item parameters.

Unfortunately, the conditional ML approach does not apply to model (7). For $k = 2$, Agresti and Lang (1993) eliminated the subject parameters by noting a corresponding model for the (m + 1) × (m + 1) table of observed counts. For the responses (X_{v1}, X_{v2}) by subject v, let

$$L_{ab} = \log\left\{\frac{p(X_{v1} > a, X_{v2} \le b)}{p(X_{v1} \le a, X_{v2} > b)}\right\}.$$

By the assumed independence of (X_{v1}, X_{v2}) each joint probability in this expression factors as the product of marginal probabilities. Hence, $L_{ab} = \text{logit}\left[p(X_{v2} \le b)\right] - \text{logit}\left[p(X_{v1} \le a)\right]$, which equals $(\theta_{vb} - \theta_{va}) - (\sigma_2 - \sigma_1)$ for model (7). Thus,

$$L_{ab} + L_{ba} = 2(\sigma_1 - \sigma_2), \tag{8}$$

for all $a \le b$. This expression applies to the (m + 1) × (m + 1) table of probabilities for each subject, and the same relationship holds for the (m + 1) × (m + 1) joint distribution $\{p_{ab}\}$ averaged over subjects; that is,

$$\log\left(\frac{\sum_{a'>a}\sum_{b'\le b} p_{a'b'}}{\sum_{a'\le a}\sum_{b'>b} p_{a'b'}}\right) + \log\left(\frac{\sum_{a'>b}\sum_{b'\le a} p_{a'b'}}{\sum_{a'\le b}\sum_{b'>a} p_{a'b'}}\right) = 2(\sigma_1 - \sigma_2) \tag{9}$$

for all $a \le b$.

Representation (9) suggests a way to estimate the difference in item parameters for the cumulative logit model (7) applied to two items. One can maximize the likelihood for the (m + 1) × (m + 1) observed table, subject to the constraint (9) holding for all m(m + 1)/2 combinations of $a \le b$. The special case with no item effect (i.e., constraining the sum of log odds to equal 0 for all $a \le b$) is an alternative characterization of symmetry. One obtains the estimated item effect using methods for maximizing a likelihood subject to constraints (Lang and Agresti, 1994). Agresti and Lang (1993) described this analysis for this model, and showed how to extend it to k items. The general case (7) corresponds to a Rasch model for all m binary collapsings of the response, with the same item effects for each collapsing. Estimated item parameters relate to those obtained by fitting a quasi-symmetry model simultaneously to all such collapsings, using the same main effect parameters for each.

Model	*Likelihood-ratio statistic*	*Pearson statistic*	*Degrees of freedom*
Mutual independence	94.9	78.5	9
Complete symmetry	378.4	282.9	6
Ordinal quasi symmetry	5.4	4.0	5
Cumulative logit	6.9	5.5	5
Quasi symmetry	2.6	2.5	3

Table 4: Goodness of fit of models for Table 2

To illustrate the cumulative logit model, we analyze Table 2. The nature of the response categories (Always wrong, Almost always wrong, Wrong only sometimes, Not wrong at all) makes the use of equally-spaced response scores questionnable, and it is not obvious what scores are appropriate. The cumulative logit model does not require such a choice. For these data (cf. Table 4), the ML fit of the model (9) used to obtain the estimated item effect for the cumulative logit model has $G^2 = 6.86$ and $X^2 = 5.46$, based on df = 5. By contrast, the symmetry model has $G^2 = 378.4$ *and* $X^2 = 282.9$ (df = 6). The ML estimate of $\sigma_2 - \sigma_1$ is 4.46 (ase = 0.43). Responses regarding teen sex tended to be much more conservative than those regarding premarital adult sex.

Similar substantive results occur in using the adjacent-categories-logit model (5) for these data. The conditional ML estimated effect is 2.628 (ase = 0.353). The estimated effect is smaller than with the cumulative logit model, since the log odds ratio refers to adjacent response categories rather than the entire scale. The related ordinal quasi-symmetry model fits well ($G^2 = 5.4$, $X^2 = 4.0$, df = 5), and also gives strong evidence that attitudes are much more conservative towards teen sex than adult sex. Table 4 describes the fit of this and other models. The ordinal models fit essentially as well as the general quasi-symmetry model, but fit much better than the mutual independence or complete symmetry models. Compared to general quasi symmetry, they have the advantage of simpler interpretation.

5. Analysis of Ordinal Matched Pairs

This section considers separately the special case $k = 2$, which occurs for matched-pairs data. In this case, quasi-symmetry models have simple logit representations, and additional ways exist of obtaining item estimates.

The logit model (5) for adjacent categories relates to a special case (6) of quasi symmetry. Letting $\sigma = \sigma_2 - \sigma_1$, that loglinear model is equivalent to the logit model

$$\log\left(p_{ab}/p_{ba}\right) = \sigma(b - a). \tag{10}$$

In fact, we can also estimate σ using software for logistic regression models, treating $\{n_{ab}, a < b\}$ as independent binomial variates with sample sizes $\{n_{ab} + n_{ba}\}$. Given that model (10) holds, marginal homogeneity is equivalent to symmetry, which is the case $\sigma = 0$.

One can base simple tests of marginal homogeneity on model (10). A Wald test uses as test statistic the ratio of $\hat{\sigma}$ to its asymptotic standard error. The likelihood-ratio test utilizes the difference between the G^2 statistics for the symmetry model and model (10). Rao's efficient score test is based on the difference in sample means for the marginal distributions, for equally-spaced category scores. Specifically, let $\{p_{ij}\}$ denote the sample proportions in the observed $(m + 1) \times (m + 1)$ table. A z test statistic is the ratio of $[\Sigma_i\, i(p_{i+} - p_{+i})]$ to its estimated standard error, which is the square root of $(1/N)\,[\Sigma_i \Sigma_j (i - j)^2 p_{ij} - d^2]$.

For cumulative logit model (7) with $k = 2$, a simple estimate of $\sigma = \sigma_2 - \sigma_1$ uses the fact that the model implies a Rasch model for each of the m collapsings of the response to a binary variable. For each collapsing, we can use the off-main-diagonal cells of the 2×2 table to get an estimate in the form of the binary conditional ML estimate for two items, $\log(n_{12}/n_{21})$. We can obtain a nearly efficient estimator by combining these, adding the numerators and adding the denominators before taking their ratio and their logarithm (Agresti and Lang 1993). In terms of the cell counts in the full $(m + 1) \times (m + 1)$ table, the resulting estimate is

$$\hat{\sigma} = \log \frac{\sum_{i<j} (j - i) n_{ij}}{\sum_{i>j} (i - j) n_{ij}}. \tag{11}$$

and the other permitting heterogeneous item effects. The related quasi-symmetry models also have homogeneous or heterogeneous main effects, with the symmetric interaction term having different parameters for each gender. Agresti (1993b) gave examples of this type.

The ordinal item-response models are rather simplistic, and the related quasi-symmetry models fit well in a limited range of situations. Even when quasi-symmetric models show lack of fit, however, they usually fit much better than complete symmetry or mutual independence loglinear models. They are designed to detect marginal shifts in location, but may fit poorly when marginal distributions show differences in dispersion as well as location. Nevertheless, the models address components of relationships not analyzed by standard loglinear analyses of associations. In practice, they should often provide useful comparisons of response distributions for ordinal items.

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