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A Hierarchical System of Interaction Measures for Multidimensional Contingency Tables

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Summary

Two summary measures are proposed of the amount of rth degree ($r+1$ factor) and higher interaction displayed in a $(d+1)$-dimensional contingency table, $1 \leqslant r \leqslant d$. The measures are based on a comparison of the "partially-raked" table having the r-factor interactions and lower-order effects removed to the "fully-raked" table in which all effects are removed. The measures have a proportional reduction in error construction with reference to using d of the variables to predict the remaining one within the partially-raked and fully-raked tables, and they are generalizations of the lambda and tau measures of association for cross-classifications of two nominal variables.

Keywords: THREE-FACTOR INTERACTION; ASSOCIATION; GOODMAN AND KRUSKAL'S TAU AND LAMBDA; MARGINAL STANDARDIZATION; RAKING

1. Introduction

In this paper we formulate two sets of summary measures of interaction for multidimensional contingency tables. For a $(d+1)$-dimensional table, these measures can be used to summarize the extent of all rth degree and higher interaction, for $1 \leqslant r \leqslant d$. An important special case is $d = 2$ and $r = 2$; that is, three-factor interaction in a three-dimensional table.

We define the measures for an adjusted table in which all interactions of lower degree have been removed but the higher-factor interactions are the same as in the original table. We refer to this table in which the r-factor interactions and all lower-order effects are absent as a "partially-raked" contingency table. If the original (unraked) table lacks $(r+1)$-factor (rth degree) or higher-factor interaction, then this adjusted table will contain identical cell entries. We refer to the table with identical cell entries, which is the transformation of the original table in which *all* effects have been removed, as the "fully-raked" table. The measures of interaction may be interpreted as summary descriptions of the discrepancy between the partially-raked table and the fully-raked table. They are defined using proportional reduction in error constructions.

Partially-raked tables lend themselves naturally to summary interaction measures because the process of removing lower-order effects clarifies the nature of the higher-order interactions but does not alter their values as expressed in terms of odds ratios. For example, the partial-raking of a two-dimensional table to obtain uniform marginal proportions is obtained through iterative adjustment of cell frequencies by constant multiplications within rows and within columns, and hence it does not affect odds ratios in 2×2 blocks of the table. Similarly, three-dimensional tables can be adjusted to obtain uniform two-dimensional marginal distributions without affecting ratios of odds ratios in $2 \times 2 \times 2$ blocks of the table.

In the next section we define measures of association for two-dimensional tables which have been raked with respect to marginal effects. In Sections 3 and 5, analogous measures are defined for describing three-factor interaction in three-dimensional tables. In Section 6, the measures are further generalized to describe rth degree and higher interaction in a $(d+1)$-dimensional table ($r \leqslant d$). Asymptotic sampling distributions are given in the final section.

In proposing these measures, we recognize the impossibility of adequately representing the interaction patterns in a large table by a single number. However, we believe that these measures help to indicate the order of magnitude of the interactions.

2. MEASURES OF ASSOCIATION FOR RAKED TWO-DIMENSIONAL TABLES

The simplest setting for our measures is a two-dimensional contingency table representing the cross-classification of a population on two variables. Let ρ_{ij} represent the proportion of members classified in the cell in row i and column j, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant c$. We consider a corresponding table of proportions $P^* = \{\rho_{ij}^*\}$ having the same odds ratios for all 2×2 subtables (i.e. $\rho_{ij}^* \rho_{rc}^* / \rho_{ic}^* \rho_{rj}^* = \rho_{ij} \rho_{rc} / \rho_{ic} \rho_{rj}$, $1 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant c-1$) yet having uniform marginal distributions $\{\rho_{i.}^* = 1/r, 1 \leqslant i \leqslant r\}$ and $\{\rho_{.j}^* = 1/c, 1 \leqslant j \leqslant c\}$. In other words, the original table has been partially-raked so as to maintain the same degree of two-factor interaction (as measured by odds ratios) while removing the one-factor effects. If the original table displays independence, then we obtain $\rho_{ij}^* = 1/rc$ (all i,j). Thus, a measure of association describes how far P^* differs from the uniform (fully-raked) table, whose probabilities we denote by $P^{**} = \{\rho_{ij}^{**}\}$ to denote the raking with respect to one-factor and two-factor effects.

Measures of the association displayed in P^* which have simple operational interpretations can be constructed using the proportional reduction in error (*PRE*) approach outlined by Goodman and Kruskal (1954). We identify one of the variables as a dependent variable Y (say, corresponding to the row classification). After selecting a rule for predicting classification on Y, we obtain the expected proportion of prediction errors when the rule is applied to predict Y within each of the levels of X, (i) according to P^*, and (ii) according to P^{**}. The measure is defined to be the *PRE* obtained by basing the predictions of Y on the interaction structure in P^* instead of in P^{**}.

Within each column, the optimal rule is to predict that each member is classified in the modal category of Y for that particular level of X. Let $\rho_{mj}^* = \max_{1 \leqslant i \leqslant r}\{\rho_{ij}^*\}$. That is, $m = m(j)$ is the row index at which the maximum proportion occurs in the jth column of the partially-raked table. Notice that $m(j)$ need not be unique, but the value of ρ_{mj}^* will be. The proportion of members who are classified in the jth column and whose value of Y is incorrectly predicted using the 'modal" prediction rule is $\rho_{.j}^* - \rho_{mj}^* = 1/c - \rho_{mj}^*$. Summing over all columns, we obtain the proportion of misclassifications when predicting Y according to the association in P^* to be

$$\sum_j (\rho_{.j}^* - \rho_{mj}^*) = 1 - \sum_j \rho_{mj}^*.$$

When Y is predicted according to P^{**}, the proportion of misclassifications is

$$\sum_j (\rho_{.j}^{**} - \rho_{mj}^{**}) = 1 - 1/r.$$

The *PRE* obtained in using the $X-Y$ association structure to predict Y is

$$\lambda^* = (r \sum_j \rho_{mj}^* - 1)/(r-1). \tag{2.1}$$

The symbol lambda is used to reflect the similarity of this measure to the nominal measure of association lambda which is based on this same prediction rule for unraked tables (see Goodman and Kruskal, 1954).

Since $\rho_{mj}^* \geqslant 1/rc$ with X and Y being independent iff all $\rho_{mj}^* = 1/rc$ (in which case all $\rho_{ij}^* = 1/rc$), the independence of X and Y is equivalent to $\lambda^* = 0$. This equivalence does not apply to the lambda measure for the unraked table. In that case, lambda equals zero whenever the modal Y response is the same for all levels of X.

The tau measure for nominal variables proposed by Goodman and Kruskal (1954), which employs a proportional prediction rule, can also be generalized to partially-raked tables. Predicting Y within levels of X according to P^*, we obtain an expected proportion of misclassifications of

$$\sum_j [\sum_i \rho_{ij}^* (1 - \rho_{ij}^*/\rho_{.j}^*)] = 1 - c \sum_i \sum_j \rho_{ij}^{*2}.$$

The *PRE* obtained by using, instead of ignoring , the $X-Y$ associations to predict Y is

$$\tau^* = (rc\sum_i\sum_j \rho_{ij}^{*2} - 1)/(r-1). \tag{2.2}$$

It is easily seen that $\tau^* = 0$ or $\tau^* = 1$ in the same cases as does λ^*. Due to the uniformity of the $\{\rho_{ij}^{**}\}$, λ^* and τ^* generate the same expected proportion of errors for the predictions based on P^{**}. In basing the predictions on the association in P^*, though, the modal prediction rule can result in no more prediction errors than are expected using the proportional prediction rule. Hence $\tau^* \leqslant \lambda^*$, so that λ^* might seem to be the preferred measure in this context. However, τ^* depends on the $\{\rho_{ij}^*\}$ through their relative sizes rather than just through their column-wise maxima, and hence it discriminates between many tables for which λ^* does not.

3. Measures of Three-factor Interaction for Three-dimensional Tables

We now generalize the approach of the previous section to define measures of the degree of three-factor interaction displayed in a three-dimensional cross-classification of variables Y, X_1, and X_2. Let ρ_{ijk} denote the proportion of members classified in level i of Y, level j of X_1 and level k of X_2 $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant c,\ 1 \leqslant k \leqslant l)$. Let $P^{**} = \{\rho_{ijk}^{**}\}$ denote the corresponding proportions for the partially-raked table in which two-factor and one-factor effects have been eliminated while preserving the three-factor interactions. That is, letting

$$\alpha_{ijk} = \rho_{ijk}\,\rho_{rck}/\rho_{ick}\,\rho_{rjk}, \quad \alpha_{ijk}^{**} = \rho_{ijk}^{**}\,\rho_{rck}^{**}/\rho_{ick}^{**}\,\rho_{rjk}^{**},$$

the $\{\rho_{ijk}^{**}\}$ satisfy

$$\alpha_{ijk}/\alpha_{ijl} = \alpha_{ijk}^{**}/\alpha_{ijl}^{**}, \quad 1 \leqslant i \leqslant r-1, \quad 1 \leqslant j \leqslant c-1, \quad 1 \leqslant k \leqslant l-1,$$

under the constraints that

$$\rho_{ij.}^{**} = 1/rc, \rho_{i.k}^{**} = 1/rl, \rho_{.jk}^{**} = 1/cl, \quad 1 \leqslant i \leqslant r, \quad 1 \leqslant j \leqslant c, \quad 1 \leqslant k \leqslant l.$$

Let P^{***} denote the corresponding fully-raked table of entries $\{\rho_{ijk}^{***} = 1/rcl\}$, for which one-, two-, and three-factor effects have been eliminated.

Now consider the *PRE* arising from predicting Y within all combinations of levels of X_1 and X_2 utilizing the structure of the three-factor interactions in P^{**} instead of P^{***}. The use of the proportional prediction rule results in a *PRE* of

$$\tau^{**} = (rcl\sum_i\sum_j\sum_k \rho_{ijk}^{**2} - 1)/(r-1). \tag{3.1}$$

Note that $\tau^{**} = 0$ iff all $\rho_{ijk}^{**} = 1/rcl$, which is equivalent to an absence of three-factor interaction in the unraked and partially-raked tables. This measure may also be interpreted as a normalization of the measure $\sum_i\sum_j\sum_k(\rho_{ijk}^{**} - 1/rcl)^2$ describing the distance between the partially-raked and fully-raked tables. Tau is symmetric in the labelling of the independent variables, and it is also symmetric in the choice of dependent variable if $r = c = l$.

The use of the modal prediction rule to make the predictions on Y using P^{**} instead of P^{***} results in a *PRE* of

$$\lambda^{**} = (r\sum_j\sum_k \rho_{mjk}^{**} - 1)/(r-1), \tag{3.2}$$

where $\rho_{mjk}^{**} = \max_i \{\rho_{ijk}^{**}\}$. Note that when $r = 2$, $\lambda^{**} = \sum_i\sum_j\sum_k |\rho_{ijk}^{**} - 1/rcl|$. Like τ^{**}, λ^{**} is symmetric in the labelling of the independent variables, though it is not necessarily invariant to the choice of dependent variable even when $r = c = l$. Generally, $\tau^{**} \leqslant \lambda^{**}$, with $\lambda^{**} = 0$ equivalent to $\tau^{**} = 0$ and $\lambda^{**} = 1$ equivalent to $\tau^{**} = 1$.

For a $2 \times 2 \times 2$ table, each entry in P^{**} equals ρ_{111}^{**} or $\frac{1}{4} - \rho_{111}^{**}$, and $\lambda^{**} = \sqrt{(\tau^{**})} = |1 - 8\rho_{111}^{**}|$. Also letting R denote the ratio of odds ratios $R = \alpha_{111}/\alpha_{112}$,

$$\lambda^{**} = |(1 - R^{\frac{1}{4}})/(1 + R^{\frac{1}{4}})|. \tag{3.3}$$

It follows that $\lambda^{**} = \tau^{**} = 1$ when $R = 0$ or $R = \infty$. The measure (3.3) was proposed by Sanathanan and Bridges (1978) to describe the degree of interaction in a $2 \times 2 \times 2$ table. In fact, for $2 \times c \times l$ tables, they suggested a measure which is equivalent to λ^{**}.

Whereas the absence of interaction is a well-defined state, the maximal degree of interaction is not. Table 1 illustrates maximal interaction as expressed by the tau and lambda measures for two different table dimensions. For the dimension $2 \times 2 \times 4$, $\tau^{**} = \lambda^{**} = 1$ when there are two sets of 2×2 tables that display complete similarity within sets and complete dissimilarity between sets; i.e. the Y–X_1 association is the strongest positive association within two levels of X_2 and the strongest negative association within the other two levels of X_2. For the dimension $2 \times 2 \times 3$, the maximal value of τ^{**} and λ^{**} occurs when the $Y - X_1$ association is the strongest positive at one level of X_2, the strongest negative at another level of X_2, and absent at the third level of X_2.

TABLE 1
Maximal interaction for $2 \times 2 \times 4$ *and* $2 \times 2 \times 3$ *cross-classifications*

$2 \times 2 \times 4$:	1/8	0	1/8	0	0	1/8	0	1/8
	0	1/8	0	1/8	1/8	0	1/8	0
$2 \times 2 \times 3$:	1/6	0	1/12	1/12	0	1/6		
	0	1/6	1/12	1/12	1/6	0		

4. EXAMPLE

A summary measure of interaction can help us to gauge the importance of interaction terms in a log-linear or logit model. In particular, there are two situations when an interaction measure definitely complements χ^2 goodness-of-fit tests for interactions. For very large sample sizes the χ^2 test might result in rejecting a model such as one containing no three-factor interaction terms, yet a measure such as τ^{**} may be very close to zero. This means that the three-factor interaction is statistically significant yet possibly weak in substantive terms. Hence, the model which ignores it might be reconsidered in the light of the particular application in order to achieve simplicity and parsimony. On the other hand, for small sample sizes a particular model may be judged adequate in a goodness-of-fit test, yet τ^{**} might be of substantial size. The measure τ^{**} would of course have a large standard error, but its size would emphasize that the acceptance of the model should be a very tentative one and that a larger sample size might show the model to be grossly inadequate.

Table 2 illustrates the first of these two cases. This table gives the joint distribution of Y = voter registration, X_1 = race and X_2 = region for a sample of adults in the United States in 1976. When Table 2 is raked with respect to all but three-factor interactions, we obtain Table 3. To get this table, we applied seven cycles of the iterative proportional fitting scheme

$$t_{ijk}^{(3s)} = t_{ijk}^{(3s-1)}/rlt_{i.k}^{(3s-1)}, \quad t_{ijk}^{(3s-1)} = t_{ijk}^{(3s-2)}/clt_{.jk}^{(3s-2)},$$
$$t_{ijk}^{(3s-2)} = t_{ijk}^{(3s-3)}/rct_{ij.}^{(3s-3)}, \quad s = 1, 2, \ldots,$$

where $\{t_{ijk}^{(0)}\}$ are the frequencies in Table 2. We note that the partially-raked table (Table 3) is very close to the fully-raked table, as reflected by the sample values $\hat{\tau}^{**} = 0{\cdot}002$ and $\hat{\lambda}^{**} = 0{\cdot}038$.

From our experience in using τ^{**} and λ^{**} with dichotomous dependent variables, values in the ranges of $\tau^{**} \leqslant 0{\cdot}01$ and $\lambda^{**} \leqslant 0{\cdot}01$ tend to occur for tables that display a relatively weak amount of three-factor interaction. For example, suppose that $|\rho_{ijk}^{**} - 1/rcl| \leqslant \delta/rcl$ (all i, j, k) for some δ, $0 \leqslant \delta \leqslant 1$. Then $\tau^{**} \leqslant \delta^2(r-1)$ and $\lambda^{**} \leqslant \delta/(r-1)$, and when $\delta = 0{\cdot}1$ with $r = 2$ we get $\tau^{**} \leqslant 0{\cdot}01$ and $\lambda^{**} \leqslant 0{\cdot}1$. Hence, the degree of three-factor interaction seems to be very weak for

TABLE 2
Three-way table for voter registration, race and region

Race	Voter registration	Region: North	North Central	South	West
White	Yes	13 827	17 457	17 151	10 125
	No	6 711	6 385	8 571	5 508
Black	Yes	946	1 299	2 985	554
	No	784	661	2 310	357
Spanish	Yes	333	170	544	606
	No	600	152	861	1 103

Source: Based on Table 2 in US Bureau of the Census (1978).
Note: These frequencies are approximations, since the referenced table contained estimated US population frequencies based on the sample survey.

TABLE 3
Table of partially-raked proportions corresponding to Table 2

Race	Voter registration	Region: North	North Central	South	West
White	Yes	0·0440	0·0397	0·0422	0·0408
	No	0·0393	0·0436	0·0411	0·0426
Black	Yes	0·0405	0·0405	0·0408	0·0449
	No	0·0428	0·0429	0·0426	0·0384
Spanish	Yes	0·0405	0·0448	0·0420	0·0394
	No	0·0429	0·0385	0·0413	0·0440

this example. However, the likelihood ratio goodness-of-fit test of the model of no three-factor interaction results in a highly significant test statistic ($\chi^2 = 31{\cdot}53$, d.f. $= 6$, $P<0{\cdot}00005$). The statistical significance is not surprising given the very large sample size ($n = 100\,000$), and we could consider using the "no three-factor interaction" model as suggested by the small values of $\hat{\tau}^{**}$ and $\hat{\lambda}^{**}$.

5. OTHER APPROACHES TO MEASURING THREE-FACTOR INTERACTION

The proportional reduction in error construction of τ^{**} and λ^{**} necessitates the identification of a dependent variable. Often one may wish to use a measure which is invariant to the choice of dependent variable. Symmetric versions of tau and lambda may be defined for this purpose, by calculating the expected errors corresponding to selecting each variable as the dependent variable one-third of the time. For tau, for example, this modification results in the measure

$$\tau_S^{**} = \frac{rcl \sum_i \sum_j \sum_k \rho_{ijk}^{**2} - 1}{3rcl/(cl+rl+rc) - 1}. \tag{5.1}$$

One should note that a value of $\lambda^{**} = \tau^{**} = 1$ is not obtainable for all table dimensions. For example, for a $2 \times c \times l$ table, the maximal possible value is one if c and l are even numbers and otherwise it is $1 - 1/S$, where S is the minimum odd dimension. The upper bound of one is also obtainable (for the asymmetric *and* symmetric versions) when $r = c = l$, but never when $r > c$ or $r > l$. Hence, if we use τ^{**} or λ^{**} to compare interactions in two samples, the tables should have the same dimensions.

Summary measures may also be defined on the original (unraked) table. For example, Bishop, Fienberg and Holland (1975, p. 330) proposed a family of interaction measures,

$$\sum_{i,j,k} |\rho_{ijk} - \rho'_{ijk}|^a / \rho^b_{ijk} (\rho'_{ijk})^c, \tag{5.2}$$

based on the discrepancies between the true proportions and proportions $\{\rho'_{ijk}\}$ which would be expected according to a particular log-linear model. Unfortunately, the upper bounds of members of this family depend on the table dimensions even when $r = c = l$. The member given by $a = 2$, $b = 0$, $c = 1$ is asymptotically equivalent to the chi-square goodness-of-fit statistic divided by the sample size and is a constant multiple of a family of measures suggested by Sakoda (1977). The index having $a = 1$, $b = 0$, $c = 0$ is perhaps the most easily interpretable member, being the total discrepancy between the two sets of proportions.

A somewhat different approach to summarizing interaction has been suggested by Davis (1975) and Clogg (1979). They take monotonic transformations of interaction parameters from log-linear models so that they fall on a -1 to $+1$ scale. Another very different approach is based on the reduction in chi-square relative to a baseline model (see Goodman, 1971). Two disadvantages of this approach are readily apparent: the dependence of the measure on the choice of baseline model, and the non-comparability of measure values across tables for which the goodness-of-fit of the baseline model varies.

Finally, one should note that these measures apply most naturally to cross-classifications of nominal variables. If any of the variables are ordinal, it may be more informative to utilize the ordinal nature in describing the interaction. Measures may be defined to correspond to appropriate models for ordinal data. To illustrate, Goodman (1979) proposed a "uniform interaction" model for ordinal variables, whereby the ratio

$$\theta_{ijk} = \frac{\rho'_{ijk}\, \rho'_{i+1,j+1,k} / \rho'_{i+1,j,k}\, \rho'_{i,j+1,k}}{\rho'_{ij,k+1}\, \rho'_{i+1,j+1,k+1} / \rho'_{i+1,j,k+1}\, \rho'_{i,j+1,k+1}}$$

of odds ratios for all adjacent collections of cells of size $2 \times 2 \times 2$ is constant. To describe how the association between two ordinal variables tends to increase (or decrease) across the levels of the third variable, we could utilize a measure such as

$$\frac{\sum_{i=1}^{r-1} \sum_{j=1}^{c-1} \sum_{k=1}^{l-1} \hat{\rho}_{ijk}\, \hat{\rho}_{i+1,j+1,k}\, \hat{\rho}_{i,j+1,k+1}\, \hat{\rho}_{i+1,j,k+1}}{\sum_{i=1}^{r-1} \sum_{j=1}^{c-1} \sum_{k=1}^{l-1} \hat{\rho}_{i,j+1,k}\, \hat{\rho}_{i+1,j,k}\, \hat{\rho}_{i,j,k+1}\, \hat{\rho}_{i+1,j+1,k+1}}. \tag{5.3}$$

If there is uniform interaction, then the population version of this measure simplifies to the common value of θ_{ijk}.

6. Measures of Interaction for Higher-dimensional Tables

The measures we have discussed can be naturally generalized to measures of $(r+1)$-factor and higher interaction in $(d+1)$-dimensional and higher-dimensional tables, for $1 \leqslant r \leqslant d$. We will illustrate for the tau *PRE* measure.

Consider a $(d+1)$-dimensional contingency table having l_o levels for the dependent variable Y and l_i levels for variable X_i ($i = 1, \ldots, d$). Let $\mathbf{i} = (i_1, \ldots, i_d)$ and let $\rho^{(r)}_{i_0\mathbf{i}}$ be the proportion in cell $(i_0, i_1, \ldots, i_d)$ for the table which has been raked with respect to r-factor interactions and all lower-order effects. Then all $\binom{d+1}{r}$ of the r-dimensional marginal tables are uniform. $P^{(r)}$ equals the fully-raked table of proportions $\{1/l_0\, l_1 \ldots l_d\}$ iff there is no f-factor interaction, for

$f = r+1, \ldots, d+1$. The proportional prediction rule has expected error

$$\sum_{\mathrm{i}} [\sum_{i_0} \rho_{i_0\mathrm{i}}^{(r)}(1-\rho_{i_0\mathrm{i}}^{(r)}/\rho_{.\mathrm{i}}^{(r)})]$$

in predicting Y within all combinations of levels of $(X_1, \ldots, X_d)$ according to the interaction structure in $P^{(r)}$, and it leads to the *PRE* measure

$$\tau^{(r)} = [l_0 \sum_{i_0} \sum_{\mathrm{i}} (\rho_{i_0\mathrm{i}}^{(r)^2}/\rho_{\mathrm{i}}^{(r)^2}) - 1]/(l_0 - 1). \tag{6.1}$$

We obtain $\tau^{(r)} = 0$ iff there is a complete absence of $(r+1)$-factor and higher-factor interaction.

Finally, these measures need not be restricted to describing *all* forms of interaction of a certain level. For example, we can obtain a partially-raked table with some subset of the r-factor interactions removed (and all lower effects removed) and compare it to the fully-raked table. Formula (6.1) can be applied to that partially-raked table to summarize the extent of remaining interactions. To illustrate, in the three-dimensional case, we might wish to describe the total extent of three-factor interactions and partial association between Y and X_1, perhaps due to a prior belief that these effects will be negligible. The partially-raked table P^* will satisfy $\rho^*_{i.k} = 1/rl$ and $\rho^*_{.jk} = 1/cl$ all i, j, k, but will maintain the Y–X_1 partial associations $\rho_{ijk}\,\rho_{rck}/\rho_{ick}\,\rho_{rjk}$, $1 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant c-1$, $1 \leqslant k \leqslant l$, and hence also the three-factor interactions.

7. Asymptotic Distribution Theory

In this section we show how to obtain asymptotic non-null variances for the interaction measures, making it possible to construct approximate confidence intervals for population values based on observations in a probability sample. Let $\boldsymbol{\rho}$ denote the vector of population proportions and let $\hat{\boldsymbol{\rho}}$ denote an estimate of $\boldsymbol{\rho}$. Let $\boldsymbol{\rho}^{(r)}$ and $\hat{\boldsymbol{\rho}}^{(r)}$ denote the corresponding population and sample proportions for a partially-raked table in which some set (possibly all) of the r-factor interactions have been eliminated, as well as all lower-order effects. Let $\boldsymbol{\xi}$ be a vector specifying the fixed marginal distributions of the partially-raked table, and let A be the matrix whose columns generate those marginal probabilities when applied to $\boldsymbol{\rho}^{(r)}$ or $\hat{\boldsymbol{\rho}}^{(r)}$. Also, let K denote an ortho-complement matrix to A which, when applied to the vectors $\ln \boldsymbol{\rho}$ and $\ln \boldsymbol{\rho}^{(r)}$, describes the interactions that are preserved in the partially-raked table.

If an estimate $\hat{\boldsymbol{\rho}}^{(r)}$ exists which satisfies the above constraints, it may be found using the iterative proportional fitting algorithm (see, for example, Bishop *et al.*, 1975, Section 3.5). Although $\hat{\boldsymbol{\rho}}^{(r)}$ does not have a closed-form expression except for special cases, its asymptotic covariance matrix can be obtained by applying implicit techniques to the expressions $A'\hat{\boldsymbol{\rho}}^{(r)} = \boldsymbol{\xi}$ and $K' \ln \hat{\boldsymbol{\rho}}^{(r)} = K' \ln \hat{\boldsymbol{\rho}}$. Let V denote the covariance matrix of $\hat{\boldsymbol{\rho}}$ and let D and D_r denote diagonal matrices with the elements of $\boldsymbol{\rho}$ and $\boldsymbol{\rho}^{(r)}$ on the diagonal. Freeman and Koch (1976) showed (though the expression is printed incorrectly in their paper) that the asymptotic covariance matrix for $\hat{\boldsymbol{\rho}}^{(r)}$ is

$$V^{(r)} = K[K'D_r^{-1}K]^{-1}K'D^{-1}VD^{-1}K[K'D_r^{-1}K]^{-1}K'. \tag{7.1}$$

In the special case of simple random sampling, $V = (D - \boldsymbol{\rho}\boldsymbol{\rho}')/n$ and the asymptotic covariance matrix simplifies to

$$V^{(r)} = K[K'D_r^{-1}K]^{-1}K'D^{-1}K[K'D_r^{-1}K]^{-1}K'/n. \tag{7.2}$$

Once $V^{(r)}$ is obtained, the asymptotic variance of a wide variety of functions of $\hat{\boldsymbol{\rho}}^{(r)}$ (such as $\hat{\tau}^{(r)}$) can be obtained using the delta method, as outlined by Goodman and Kruskal (1963). Let the vector of partial derivatives of a measure $\hat{\zeta} = \hat{\zeta}(\hat{\boldsymbol{\rho}}^{(r)})$ taken with respect to $\hat{\boldsymbol{\rho}}^{(r)}$ and evaluated at $\boldsymbol{\rho}^{(r)}$ be denoted by $\partial\boldsymbol{\zeta}$. Then the asymptotic variance of $\hat{\zeta}$ is $\sigma^2(\hat{\zeta}) = \partial\boldsymbol{\zeta}'\,V^{(r)}\,\partial\boldsymbol{\zeta}$. Under simple random sampling, the asymptotic normality of $\hat{\boldsymbol{\rho}}$ induces an asymptotic normal distribution for

$\hat{\boldsymbol{\rho}}^{(r)}$. It follows that

$$(\hat{\zeta}-\zeta)/\hat{\sigma}(\hat{\zeta}) \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty,$$

where

$$\hat{\sigma}(\hat{\zeta}) = \{\widehat{\partial \boldsymbol{\zeta}}' K[K'\hat{D}_r^{-1} K]^{-1} K'\hat{D}^{-1} K[K'\hat{D}_r^{-1} K]^{-1} K'\widehat{\partial \boldsymbol{\zeta}}/n\}^{\frac{1}{2}} \tag{7.3}$$

and $\widehat{\partial \boldsymbol{\zeta}}$, $\hat{D}_r$. and $\hat{D}$ are consistent estimates of $\partial \boldsymbol{\zeta}$, D_r, and D obtained by replacing $\boldsymbol{\rho}^{(r)}$ and $\boldsymbol{\rho}$ by their consistent estimates $\hat{\boldsymbol{\rho}}^{(r)}$ and $\hat{\boldsymbol{\rho}}$.

To illustrate the asymptotic theory we first consider a two-dimensional table which has been raked with respect to single-factor effects. The matrix K here reflects the stability of the odds ratios in $\boldsymbol{\rho}$ and $\boldsymbol{\rho}^*$ through the contrasts

$$\ln \rho_{ij} - \ln \rho_{ic} - \ln \rho_{rj} + \ln \rho_{rc} = \ln \rho^*_{ij} - \ln \rho^*_{ic} - \ln \rho^*_{rj} + \ln \rho^*_{rc}, \quad 1 \leqslant i \leqslant r-1, \quad 1 \leqslant j \leqslant c-1.$$

That is, K' is a $(r-1)(c-1)\times rc$ matrix satisfying $K'\ln \boldsymbol{\rho}^* = K'\ln \boldsymbol{\rho}$. Let $\sigma_{ij,i'j'} = \operatorname{cov}(\hat{\rho}^*_{ij}, \hat{\rho}^*_{i'j'})$ as obtained from $V^{(1)}$ as given in (7.2). It follows that the asymptotic non-null variance of $\hat{\tau}^*$ under random sampling is

$$\sigma^2(\hat{\tau}^*) = [2rc/(r-1)]^2 \sum_{i,j} \sum_{i',j'} \rho^*_{ij}\, \rho^*_{i'j'}\, \sigma_{ij,i'j'}. \tag{7.4}$$

Similarly, for the random sample analogue $\hat{\lambda}^*$ of λ^*, assuming $m = m(j)$ is unique for all j, we obtain

$$\sigma^2(\hat{\lambda}^*) = [r/(r-1)]^2 \sum_j \sum_{j'} \sigma_{m(j)j,\, m(j')j'}. \tag{7.5}$$

Next consider a three-dimensional table obtained by raking with respect to all two-factor and single-factor effects. The matrix K here reflects the common ratios of odds ratios in the partially-raked and unraked tables as equated through the contrasts

$$\begin{aligned} &\ln \rho_{ijk} - \ln \rho_{ijl} - \ln \rho_{ick} + \ln \rho_{icl} - \ln \rho_{rjk} + \ln \rho_{rjl} + \ln \rho_{rck} - \ln \rho_{rcl} \\ &\quad = \ln \rho^{**}_{ijk} - \ln \rho^{**}_{ijl} - \ln \rho^{**}_{ick} + \ln \rho^{**}_{icl} - \ln \rho^{**}_{rjk} + \ln \rho^{**}_{rjl} + \ln \rho^{**}_{rck} - \ln \rho^{**}_{rcl}, \\ &\qquad\qquad 1 \leqslant i \leqslant r-1, \quad i \leqslant j \leqslant c-1, \quad 1 \leqslant k \leqslant l-1. \end{aligned}$$

That is, K' is a $(r-1)(c-1)(l-1)\times rcl$ matrix satisfying $K'\ln \boldsymbol{\rho}^{**} - K'\ln \boldsymbol{\rho}$.

Given the asymptotic covariance matrix (7.2) for $\boldsymbol{\rho}^{**}$, we can obtain asymptotic variances for the sample measures of three-factor interaction. More generally, the asymptotic variance of $\hat{\tau}^{(r)}$ or $\hat{\lambda}^{(r)}$ for $(d+1)$-dimensional cross-classifications (see (6.1)) may be obtained. For example, letting

$$\sigma_{i_0\mathbf{i},\, j_0\mathbf{j}} = \operatorname{cov}(\hat{\rho}^{(r)}_{i_0\mathbf{i}}, \hat{\rho}^{(r)}_{j_0\mathbf{j}}),$$

we obtain

$$\sigma^2(\hat{\tau}^{(r)}) = [l_0/(l_0-1)]^2 \sum_{i_0,\mathbf{i}} \sum_{j_0,\mathbf{j}} \sigma_{i_0\mathbf{i}\, j_0\mathbf{j}} \times \left[\frac{2\rho^{(r)}_{i_0\mathbf{i}}\, \rho^{(r)}_{.\mathbf{i}} - (\sum_{i_0'} \rho^{(r)2}_{i_0'\mathbf{i}})}{\rho^{(r)2}_{.\mathbf{i}}}\right]\left[\frac{2\rho^{(r)}_{j_0\mathbf{j}}\, \rho^{(r)}_{.\mathbf{j}} - (\sum_{j_0'} \rho^{(r)2}_{j_0'\mathbf{j}})}{\rho^{(r)2}_{.\mathbf{j}}}\right] \tag{7.6}$$

if $r<d$. When $r = d$ (such as with τ^{**} for three-dimensional tables), $\tau^{(r)}$ simplifies to

$$[l_0\, l_1 \ldots l_d \sum_{i_0} \sum_{\mathbf{i}} \rho^{(r)2}_{i_0\mathbf{i}} - 1]/(l_0-1),$$

and we get

$$\sigma^2(\hat{\tau}^{(d)}) = [2l_0 \ldots l_d/(l_0-1)]^2 \sum_{i_0,\mathbf{i}} \sum_{j_0,\mathbf{j}} \rho^{(r)}_{i_0\mathbf{i}}\, \rho^{(r)}_{j_0\mathbf{j}}\, \sigma_{i_0\mathbf{i},\, j_0\mathbf{j}}. \tag{7.7}$$

The K matrix that helps determine $V^{(r)}$ in the case $r = 2$ describes the

$$l_0 \ldots l_d - \left\{1 + \sum_{i=0}^{d} (l_i - 1) + \sum_{i<j} (l_i - 1)(l_j - 1)\right\}$$

higher-order interactions that are preserved in raking the $(d+1)$-dimensional table with respect to one-factor and two-factor effects.

For the special case of the $2 \times 2 \times 2$ table, expression (7.7) simplifies to

$$\sigma^2(\hat{\tau}^{**}) = 256[\sum \rho_i^{-1}](8\rho_{111}^{**} - 1)^2[\rho_{111}^{**}(\tfrac{1}{4} - \rho_{111}^{**})]/n \qquad (7.8)$$

Similar expressions can easily be given for the asymptotic variance of $\hat{\lambda}^{(r)}$ or of symmetric versions of these measures.

We computed $\hat{\sigma}(\hat{\tau}^{**}) = 0{\cdot}0009$ and $\hat{\sigma}(\hat{\lambda}^{**}) = 0{\cdot}0077$ for the standard errors of $\hat{\tau}^{**} = 0{\cdot}0020$ and $\hat{\lambda}^{**} = 0{\cdot}0384$ used to describe three-factor interaction in Table 2. Hence, the true interaction estimated from that table may be concluded to be non-zero, but very weak.

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