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Considerations in Measuring Partial Association for Ordinal Categorical Data

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Some of the measures commonly used to describe association between two ordinal categorical variables, controlling for the effects of other ordinal categorical variables, are compared in their closeness to a corresponding conditional measure for an underlying jointly continuous distribution. Partial gamma and Kendall's partial tau are observed to be poorer than a weighted average of the Kendall's τ_b's from separate control levels, according to this criterion. The asymptotic sampling distribution of a particular weighted average measure is derived, and the behavior of the mean and asymptotic variance of such a measure is analyzed as the categorical variables are more finely measured.

KEY WORDS: Ordinal measures; Partial association; Partial tau; Partial gamma; Weighted average measures; Asymptotic normality.

1. INTRODUCTION

Ordinal categorical data occur frequently in social and behavioral research, where dependent, independent, and control variables may in practice be measured in classes (such as high, medium, low or above median, below median) or by scales such as the Bogardus social distance scale or Likert type scales. In this paper, we shall consider methods for measuring the association between two categorical ordinal variables, controlling for the effects of one or more other categorical ordinal variables. Many proposals have been given for obtaining such a general summary measure, but there seems to be little agreement among practitioners in choosing among them (see Quade [12] for a summary).

There are four different settings for each of the variables in which the various measures of association could be computed. The phenomena under consideration could exist at a continuous level or at a discrete level. Those existing at a continuous level could be measured in the sample in a continuous or discrete manner. Those which exist at a discrete level could be measured according to the same discrete levels, or could be even more crudely grouped. In analyzing these measures of association, we shall assume that the variables are inherently continuous, but are measured in the sample using ordered categories. In many applications involving ordinal variables, it would seem reasonable to imagine the existence of an underlying continuum, even though crude or underdeveloped measurement or grouping results in sample categorization.

The behavior of each of the coefficients based on discretely measured variables is considered as the measurement becomes "more continuous;" i.e., as the categorization of each of the variables is refined. To distinguish between the grouped and ungrouped versions of the same measure of association, we shall conventionally attach a superscript or subscript u to any symbol which represents a particular characteristic of the underlying continuous system of variables.

The term partial measure of association will be broadly used to refer to any descriptive measure of the degree of association between two variables X and Y, controlling for a third variable Z. Thus, the evaluation of a particular ordinal partial measure of association is dependent upon how "controlling for a variable" is interpreted. Quade [12] describes four interpretations of this term which are prevalent in the statistical and social science literature. The measures affiliated with the two most commonly used concepts of control are those based on a formulation of a linear model for sign scores for all pairs of observations, and those based on weighted averages of bivariate ordinal measures which are computed within subgroups of observations according to the Z classification.

The primary measure corresponding to the first of these concepts is the Kendall's [8] partial tau measure for ranked data. In its (ungrouped) sample form, this measure is calculated as

$$t_{XY\cdot Z}{}^u = \frac{t_{XY}{}^u - t_{XZ}{}^u t_{YZ}{}^u}{[(1 - (t_{XZ}{}^u)^2)(1 - (t_{YZ}{}^u)^2)]^{\frac{1}{2}}} , \quad (1.1)$$

where $t_{XY}{}^u$, $t_{XZ}{}^u$, and $t_{YZ}{}^u$ represent the sample values of the ordinary pairwise Kendall's taus. This measure is a special case of the Pearson partial correlation applied to the set of $(-1, 1)$ sign scores for differences of ranks on X, Y, and Z for all of the pairs of measurements. The analogous linear model for these pair scores is described in detail in [11]. For ungrouped data, this measure is also obtained by calculating the well-known coefficient

$$\phi_u = (ad - bc)/[(a + b)(b + d)(c + d)(a + c)]^{\frac{1}{2}} \quad (1.2)$$

for the following tabulation which provides a summary of concordant and discordant relationships for all pairs.

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	(Y, Z) *Concordant*	(Y, Z) *Discordant*
(X, Z) Concordant	a	b
(X, Z) Discordant	c	d

When the variables are categorized, the recent trend for many social science methodologists has been to use (1.1) and the corresponding formulas for higher order partial measures with the substitution of the t_b sample values of the Kendall's τ_b pairwise measure for categorical ordinal data (see, e.g., [7, 11]). These measures follow from the application of the linear model to $(-1, 0, 1)$ pair scores. Another related approach [16] is to select just those pairs of measurements which are untied on all of the variables, and use the measure $\hat{\phi}$. One could then apply the standard formulas for higher order partial correlations to these partial $\hat{\phi}$'s to obtain higher order partial measures.

The other commonly used family of partial measures for ordinal data is based on the concept of holding the control variable constant. Under this notion, an ordinal categorical measure $\hat{\theta}_k$ is computed for each of the c levels of a categorical control variable, and a summary partial measure is formed by taking some weighted average of these,

$$\bar{t} = \sum_{k=1}^{c} r_k \hat{\theta}_k \ . \tag{1.3}$$

The best known of these is partial gamma [4], for which $\hat{\theta}_k$ is the ordinal measure gamma for category k, and r_k equals the probability that a pair of observations tied with respect to the control variable and untied with respect to the two noncontrol variables is in category k of the control variable. Similarly, one could form some weighted average of a tau type measure such as t_a, t_b, t_c [8], d_{YX} [11], or of a Spearman correlation analog for contingency tables, or compute one of these measures for standardized tables such as the ones suggested by Rosenberg [14] and by Smith [15]. Control is commonly administered in social science research through analysis of subgroups (see, e.g., [10]), and weighted average measures are natural summarizations for such analyses. Quade [12] has defined an extension to the family of measures (1.3) based on various definitions of nearly constant values of possibly several control variables. For a single categorized control variable, his index of matched correlation includes, as a special case, measures such as partial gamma and weighted t_a.

We shall compare the behaviors of a few of the most commonly used partial categorical ordinal measures, as the measurements become finer, in Section 2. An additional partial measure of association $\bar{t}_b$ is proposed in Section 3 based on a certain weighted average of Kendall's t_b's, and its asymptotic sampling distribution is derived. It is shown that $\bar{t}_b$ is likely to be superior to partial gamma and analogues of $t_{XY \cdot Z}{}^u$ in terms of closeness to a corresponding measure of conditional association, if there is an underlying continuous system of variables. The importance of the choice of the number of categories of the control variable on the population value of the measure and the variance of the sample value is considered in the last section.

Throughout, we shall assume that there is no interaction, in the rather imprecise sense that the underlying degree of association between the two variables remains the same at each fixed single value of the underlying control variable, so that it is sensible to compute a summary descriptive partial measure of this degree of association. In many places, precise mathematical statements of the conditions needed for the convergence arguments to apply are omitted, since they would detract from the main emphasis of the paper and since they could be fulfilled for the underlying continuous models typically used in practice, such as joint normality. (See [2] for a more thorough treatment of this aspect.)

2. COMPARING ORDINAL PARTIAL MEASURES

In this section, we shall deal with the properties of the population values of the ordinal categorical measures defined in Section 1. The reason for this is that the characteristics of the population measures need to be well understood before one can decide to routinely use sample values of that measure. We denote the values of $t_{XY \cdot Z}$, $\hat{\phi}$, and $\bar{t}$ for the categorized form of the underlying joint population distribution by $\tau_{XY \cdot Z}$, ϕ, and $\bar{\tau} = \sum \rho_k \theta_k$. The probabilities of this grouped distribution are denoted by

$$p_{ijk} = P(X \in A_{ai},\ Y \in B_{bj},\ Z \in C_{ck}) \ , \tag{2.1}$$

where $\{A_{ai}\}$, $\{B_{bj}\}$, and $\{C_{ck}\}$ are partitions of the possible values of X, Y, and Z, with a, b, and c categories, respectively.

A criterion is now needed for the comparison of the various ordinal partial measures of association. Assuming no interaction in the underlying continuous system, one easily interpretable and hence desirable property would be as follows: the categorical partial measure of association based on a particular bivariate ordinal measure of association is close to the value of that ordinal measure obtained for the joint distribution of (X, Y) at a fixed single value of Z, for the underlying continuous system. We shall refer to this value for this underlying jointly continuous conditional distribution as the *underlying conditional association.*

For example, $\tau_{XY \cdot Z}$, ϕ, and weighted averages of γ, τ_a, τ_b, τ_c, or d_{YX} are all fundamentally based on the difference between the probabilities of concordant and discordant pairs of observations on the variables. Hence, they should ideally be close to the value of this difference of probabilities, namely,

$$\begin{aligned} \tau_u(X, Y | Z) = {} & P[(X_i - X_j)(Y_i - Y_j) > 0 | Z] \\ & - P[(X_i - X_j)(Y_i - Y_j) < 0 | Z] \ , \end{aligned} \tag{2.2}$$

expected for two pairs (X_i, Y_i) and (X_j, Y_j) chosen at random from the underlying jointly continuous conditional distribution at any specific single value of Z. We use the symbol τ in (2.2) to recall that Kendall's definition [8, p. 124] of the bivariate analog is precisely this difference. In accord with the concept of interaction in this article, this underlying conditional difference between the probabilities of concordant and discordant pairs of measurements is assumed to be the same at each single value of Z. Similarly, if one compared partial measures which are analogous to (1.1) and (1.3) but based on pairwise Spearman's rhos, the underlying conditional association would be the value of rho for this conditional distribution in the underlying continuous system.

Let us consider now, in more detail, the various ordinal categorical partial measures for which the underlying conditional association is $\tau_u(X, Y|Z)$. Note that interaction may exist in the grouped distribution, and the extent of it depends on the categorization of the control variable. One would expect the performance of a categorical partial measure to improve, though, in terms of approximating $\tau_u(X, Y|Z)$, as the number of categories c of the control variable increases.

As an example, suppose that (X, Y, Z) has a trivariate normal distribution, with X and Y spuriously related. That is, $\rho_{XY}{}^u = \rho_{XZ}{}^u\rho_{YZ}{}^u$ for the underlying distribution, and hence $\rho_{XY\cdot Z}{}^u = 0$. For this normal model, $\rho_{XY\cdot Z}{}^u$ is also the Pearson correlation between X and Y at a fixed single value of Z, so that from [8, p. 126]

$$\tau_u(X, Y|Z) = (2/\pi)\sin^{-1}\rho_{XY\cdot Z}{}^u = 0 \ .$$

In particular, suppose that the underlying Pearson correlations are $\rho_{XY}{}^u = .64$ and $\rho_{XZ}{}^u = \rho_{YZ}{}^u = .80$, and suppose (for simplicity) that the distribution is categorized in the form of a $2 \times 2 \times c$ table, with

$$P(X \in A_{2i}) = P(Y \in B_{2i}) = .5 \ , \quad i = 1, 2 \ .$$

Table 1 lists the corresponding values of $\tau_{XY\cdot Z}$, ϕ, partial gamma ($\bar{\gamma}$), an average of τ_b's ($\bar{\tau}_b$) weighted by $\{p_{..k}\}$, and another weighted average of τ_b's ($\bar{\tau}_b$) to be introduced in Section 3. Notice that neither $\tau_{XY\cdot Z}$ nor ϕ appears to approach the underlying conditional association value of 0 as c increases.

1. Values of Partial Measures of Association for Various 2 × 2 × C Tables with an Underlying Trivariate Normal Distribution [a]

c	$P(Z\in C_{ci})$	ϕ	$\tau_{XY\cdot Z}$	$\bar{\gamma}$	$\bar{\tau}_b$	$\bar{\tau}_b$
1	(1)	.443	—	.740	.443	.443
2	(.1,.9)	−.006	.377	.666	.341	.380
	(.5,.5)	.137	.143	.375	.144	.144
3	(.3,.4,.3)	.122	.141	.196	.075	.083
4	(.2,.3,.3,.2)	.127	.148	.129	.042	.053
5	(.2,.2,.2,.2,.2)	.138	.153	.069	.023	.027
7	(.1,.1,.2,.2,.2,.1,.1)	.137	.167	.062	.012	.025
10	(.1 each)	.163	.179	.018	.002	.007

[a] $\rho_{XY} = .64$ and $\rho_{XZ} = \rho_{YZ} = .80$.

In fact, even in the continuous case for the trivariate normal distribution, $\tau_{XY\cdot Z}{}^u = \phi_u$ is not equal to zero when $\rho_{XY\cdot Z}{}^u$ (and hence $\tau_u(X, Y|Z)$) equals zero, except for trivial cases. In that situation,

$$\tau_{XY\cdot Z}{}^u = \frac{(2/\pi)\sin^{-1}\rho_{XY}{}^u - (2/\pi)(\sin^{-1}\rho_{XZ}{}^u)(2/\pi)(\sin^{-1}\rho_{YZ}{}^u)}{\{(1-[(2/\pi)\sin^{-1}\rho_{XZ}{}^u]^2)(1-[(2/\pi)\sin^{-1}\rho_{YZ}{}^u]^2)\}^{\frac{1}{2}}}, \tag{2.3}$$

by virtue of the functional relationship between Kendall's tau and the Pearson correlation for the bivariate normal distribution. Using differential calculus, it can be shown from (2.3) that for the spurious normal system (i.e., $\rho_{XY\cdot Z}{}^u = 0$),

$$\tau_{XY\cdot Z}{}^u > 0 \ , \quad \text{if } \rho_{XZ}{}^u > 0 \text{ and } \rho_{YZ}{}^u > 0 \ , \text{ or if } \rho_{XZ}{}^u < 0 \text{ and } \rho_{YZ}{}^u < 0 \ ;$$

$$\tau_{XY\cdot Z}{}^u < 0 \ , \quad \text{if } \rho_{XZ}{}^u > 0 \text{ and } \rho_{YZ}{}^u < 0 \ , \text{ or if } \rho_{XZ}{}^u < 0 \text{ and } \rho_{YZ}{}^u > 0 \ ;$$

$$\tau_{XY\cdot Z}{}^u = 0 \ , \quad \text{only if } \rho_{XZ}{}^u \text{ or } \rho_{YZ}{}^u \text{ equals } -1, 0, \text{ or } 1 \ .$$

In the categorical setting, $\tau_{XY\cdot Z}$ or ϕ for an $a \times b \times c$ table converge in the limit to (2.3) for the continuous case, instead of $\tau_u(X, Y|Z)$, as a, b, and c increase such that $\{A_{ai}\}$, $\{B_{bj}\}$, and $\{C_{ck}\}$ get finer in an appropriate manner. Thus, for the trivariate normal system (and more generally), $\tau_{XY\cdot Z}$ or ϕ is not the same as or necessarily close to the underlying conditional association. In particular, $t_{XY\cdot Z}$ or $\hat{\phi}$ for continuous or categorical data, even for limitless sample sizes, would fail to detect spuriousness in an underlying trivariate normal population. This also implies that the analog of a partial regression coefficient (Somers' partial d) used in the general ordinal linear model for pair scores (see [7, 11]) is nonzero for the normal spurious system, since it is zero if and only if the corresponding Kendall's partial tau is zero.[1]

Measures of partial association based on holding the control variable constant represent an averaging of the degrees of association between X and Y over restricted ranges of Z values. Hence, by virtue of their construction, their failure to detect spurious association is nowhere near as severe, at least when the control variable is measured finely enough. As the categorization for Z is made uniformly finer, for the spurious normal example, partial measures such as $\bar{\gamma}$ and $\bar{\tau}_b$ converge to zero, the underlying conditional association (see Table 1). More generally, as the measurement of Z is refined, a $\bar{\tau}$ weighted average type measure converges to the expected value (with respect to the limiting distribution of the weighting scheme imposed on the categories of the control variable) of the ordinal categorical measure upon which it is based, measured for the given (X, Y) categorization. As all three variables are measured more finely, $\bar{\tau} \to \tau_u(X, Y|Z)$, for those weighted average measures for which $\tau_u(X, Y|Z)$ is the underlying conditional association. Of course, it should be realized that refinement

[1] The possibility that $t_{XY\cdot Z}$ could fail to detect what would be interpreted as spurious association for higher level measurement was also pointed out recently by Kim [9]. For some artificial three variable causal chains, he showed that the Pearson partial correlation would be zero, but concluded (p. 274) that "... ordinal partials do not allow us to detect even the simplest underlying causal structures."

of the categorizations of X and Y alone does not compensate for crude measurement of Z, since the joint distribution of (X, Y) over a broad range of Z values may be very different from the joint distribution at a single Z value. (See [2] for details on these convergence arguments.)

Kim [9] notes that weighted average measures such as $\bar{\tau}_b$ seem to be better than $\tau_{XY \cdot Z}$ in detecting spuriousness. He concludes (p. 279) that

> ... the notion of control is most clearly demonstrated by the traditional subgroup analysis; therefore, partials defined in this manner have the most clear interpretation... Until better measures of partial ordinal association are developed, one may consider using this type of partial whenever rankings contain enough ties.

In summary, if one interprets "ordinal partial association" to mean "bivariate ordinal association between X and Y at a fixed single value of Z for the underlying continuous distribution (when there is no interaction)," then weighted average type measures would be considered superior to $\tau_{XY \cdot Z}$ or ϕ in terms of measuring the degree of this association as finer measurement is achieved. Also, as researchers become more familiar with them, ordinal measures of association are likely to be more commonly used for casual modeling in the social sciences. For such purposes, weighted average partial measures would seem to be more reliable in terms of leading to conclusions which are consistent with those which would be made if the measurement were accomplished at a higher level.

If the researcher believes that the underlying continuous variables are jointly normally distributed, he may be more interested in a summary partial measure such as $\rho_{XY \cdot Z}{}^u$ than in the underlying conditional (ordinal) association $\tau_u(X,Y|Z)$. In that case, $\tau_u(X,Y|Z) = (2/\pi)\sin^{-1}\rho_{XY \cdot Z}{}^u$. Thus, $\rho_{XY \cdot Z}{}^u$ could be approximated by $\sin[(\pi/2)\bar{\tau}_b]$ when the observed data are ordinal categorical, and the approximation would improve as measurement of the variables is refined.

Of course, alternative concepts of control lead to different preferred ordinal partial measures. For example, Somers [16, 17] believes that if the control variable is ordinal, the summary partial measure should incorporate some aspect of the ordering of the control levels. He shows that the partial (Somers) d regression coefficient contains components summarizing information lying between and within the conditional distributions for the control variable. The between component is treated as irrelevant for the weighted average measures, and they could be used even if the control variable is nominal in level.[2]

3. AN ALTERNATIVE WEIGHTED τ_b MEASURE

Consider a $\bar{\tau}$ type measure of partial association for which the underlying conditional association is $\tau_u(X, Y|Z)$. The inadequacy of $\bar{\tau}$ in measuring the underlying conditional association is due to the bias introduced by grouping Z, and the fact that a categorical two-way generalization of Kendall's tau need not produce the same value as Kendall's tau for continuous variables. Roughly speaking, the first type of bias can be reduced by increasing c, and the second type of bias can be reduced by increasing a and b and by basing $\bar{\tau}$ on a measure which is relatively stable to choice of cross classification in terms of approximating Kendall's tau for continuous data.

Unfortunately, the most popular measure of this type, partial gamma, is likely to be very poor in the last respect in most situations, especially when a and b are small (see [1, 13]). For example, notice in Table 1 that for $c = 1$, $\bar{\gamma} = .74$ whereas $\tau_{XY}{}^u = (2/\pi)\sin^{-1}\rho_{XY}{}^u = .443$, the limiting value of $\bar{\gamma}$ when X and Y are measured continuously. Of the many ordinal measures available, τ_b seems to be least susceptible to this type of bias [1], suggesting a partial measure of the form

$$\sum_{k=1}^{c} \rho_k \tau_b(\{A_{ai}\}, \{B_{bj}\}|C_{ck}) \ .$$

where $\tau_b(\{A_{ai}\}, \{B_{bj}\}|C_{ck})$ denotes the value of τ_b for a given (X, Y) cross classification conditional on Z in C_{ck}.

This measure has a clear interpretation, since each $\tau_b(\{A_{ai}\}, \{B_{bj}\}|C_{ck})$ can be considered an approximation to the underlying $\tau(X, Y|C_{ck})$, the value of Kendall's tau between X and Y for the continuous (X, Y) distribution, when Z is in C_{ck}; this, in turn, itself is close to $\tau_u(X,Y|Z)$ if C_{ck} is small. Reynolds [13] used the weighting schemes $\{\rho_k = 1/c\}$, $\{\rho_k = p_{..k}\}$, and $\{\rho_k = p_{..k}{}^2/\sum_h p_{..h}{}^2\}$ (the probability that a random pair of measurements tied on the control variable falls in C_{ck}) in some simulation studies with the trivariate normal model, and reported similar results with each. Let

$$P_{ij\cdot k}{}^{(s)} = \sum_{i'>i}\sum_{j'>j} p_{i'j'k} + \sum_{i'<i}\sum_{j'<j} p_{i'j'k}$$

and

$$P_{ij\cdot k}{}^{(d)} = \sum_{i'>i}\sum_{j'<j} p_{i'j'k} + \sum_{i'<i}\sum_{j'>j} p_{i'j'k} \ . \tag{3.1}$$

Then

$$P_k{}^{(s)} = \sum_i\sum_j p_{ijk}P_{ij\cdot k}{}^{(s)} \ , \quad P_k{}^{(d)} = \sum_i\sum_j p_{ijk}P_{ij\cdot k}{}^{(d)}$$

are the probabilities that a pair of observations are in C_{ck} and are concordant and discordant, respectively, and let

$$P_s - P_d = \sum_k (P_k{}^{(s)} - P_k{}^{(d)}) \ .$$

Davis' partial gamma is

$$\bar{\gamma} = \frac{P_s - P_d}{P_s + P_d} = \sum_k \left(\frac{P_k{}^{(s)} + P_k{}^{(d)}}{P_s + P_d}\right)\frac{P_k{}^{(s)} - P_k{}^{(d)}}{P_k{}^{(s)} + P_k{}^{(d)}} \cdot \tag{3.2}$$

Also

$$\begin{aligned} \tau_b(\{A_{ai}\}, \{B_{bj}\}|C_{ck}) &= (1/p_{..k}{}^2)(P_k{}^{(s)} - P_k{}^{(d)})/ \\ &\quad [(1 - \sum_i (p_{i.k}/p_{..k})^2)(1 - \sum_j (p_{.jk}/p_{..k})^2)]^{\frac{1}{2}} \\ &= (P_k{}^{(s)} - P_k{}^{(d)})/[\Delta_{1k}\Delta_{2k}]^{\frac{1}{2}} \ , \end{aligned} \tag{3.3}$$

[2] The reader is referred to articles by Quade [12] and Somers [16] for discussions about the difficulty of choosing one completely representative ordinal partial measure to satisfy all concepts of control.

where

$$\Delta_{1k} = p_{\cdot\cdot k}^2 - \sum_i p_{i\cdot k}^2 \ , \quad \Delta_{2k} = p_{\cdot\cdot k}^2 - \sum_j p_{\cdot jk}^2 \ . \quad (3.4)$$

The weight $(P_k^{(s)} + P_k^{(d)})/(P_s + P_d)$ is a natural one to attach to the kth control table in forming a partial measure for gamma, which ignores all pairs tied with respect to the X or Y classification. Similarly, a natural weight to choose for a partial measure based on $\{\tau_b(\{A_{ai}\}, \{B_{bj}\} | C_{ck}), 1 \leq k \leq c\}$, which uses

$$[(1 - \sum_i (p_{i\cdot k}/p_{\cdot\cdot k})^2)(1 - \sum_j (p_{\cdot jk}/p_{\cdot\cdot k})^2)]^{\frac{1}{2}}$$

to adjust for the effect of (X, Y) cross-classification ties in the kth table, would be $(\Delta_{1k}\Delta_{2k})^{\frac{1}{2}}/\sum_h (\Delta_{1h}\Delta_{2h})^{\frac{1}{2}}$. That is, let

$$\begin{aligned}\bar{\tau}_b &= \sum_k [(\Delta_{1k}\Delta_{2k})^{\frac{1}{2}}/\sum_h (\Delta_{1h}\Delta_{2h})^{\frac{1}{2}}]\tau_b(\{A_{ai}\}, \{B_{bj}\} | C_{ck}) \\ &= \sum_k (P_k^{(s)} - P_k^{(d)})/\sum_k (\Delta_{1k}\Delta_{2k})^{\frac{1}{2}} \\ &= (P_s - P_d)/\sum_k (\Delta_{1k}\Delta_{2k})^{\frac{1}{2}} \ . \quad (3.5)\end{aligned}$$

In the special case that the adjustment factor for ties is the same in each control table, the kth weight $(\Delta_{1k}\Delta_{2k})^{\frac{1}{2}}/\sum_h (\Delta_{1h}\Delta_{2h})^{\frac{1}{2}}$ reduces to $p_{\cdot\cdot k}^2/\sum_h p_{\cdot\cdot h}^2$. Thus, this weighting scheme may be considered to be an extension of the weighting scheme $\{p_{\cdot\cdot k}^2/\sum_h p_{\cdot\cdot h}^2\}$ which further adjusts for ties with respect to the (X, Y) cross classification within each control level.

Table 2 shows the weights for $\bar{\tau}_b$ and for $\bar{\gamma}$ for the spurious normal relationship considered in Section 2 when $c = 10$ and $P(Z \in C_{10,k}) = .1$ for all k. The weights would each be .10 for the weighting schemes used by Reynolds [13]. Notice that $\bar{\gamma}$ penalizes control tables more heavily than $\bar{\tau}_b$ for having ties on the (X, Y) cross classifications. On the other hand, the weighting schemes $\{1/c\}$, $\{p_{\cdot\cdot k}\}$, and $\{p_{\cdot\cdot\cdot k}^2/\sum_h p_{\cdot\cdot h}^2\}$ give no consideration to such ties as having the effect of reducing the total information available for measuring partial association. Naturally, if there is little variability in the $\{\tau_b(\{A_{ai}\}, \{B_{bj}\} | C_{ck}), 1 \leq k \leq c\}$, each of the weighting schemes produces about the same result for the summary measure.

An important benefit of using the weights as defined in $\bar{\tau}_b$ is that the asymptotic sampling distribution of the random sample version of $\bar{\tau}_b$ is much simpler to derive than that of other weighted averages of the $\{\tau_b(\{A_{ai}\}, \{B_{bj}\} | C_{ck})\}$, and of partial measures such as $\tau_{XY\cdot Z}$ and ϕ. Let

$$\phi_{ijk} = \frac{2(P_{ij\cdot k}^{(s)} - P_{ij\cdot k}^{(d)})}{\sum_h (\Delta_{1h}\Delta_{2h})^{\frac{1}{2}}} - \frac{(P_s - P_d)[\Delta_{2k}(p_{\cdot\cdot k} - p_{i\cdot k}) + \Delta_{1k}(p_{\cdot\cdot k} - p_{\cdot jk})]}{(\Delta_{1k}\Delta_{2k})^{\frac{1}{2}}[\sum_h (\Delta_{1h}\Delta_{2h})^{\frac{1}{2}}]^2} . \quad (3.6)$$

Then if $\bar{t}_b$ is the random sample of size n version of $\bar{\tau}_b$ (i.e., p_{ijk} is replaced by its maximum likelihood estimate $\hat{p}_{ijk} = n_{ijk}/n$), it is shown in the Appendix that

$$\sqrt{n}(\bar{t}_b - \bar{\tau}_b)/\sigma \xrightarrow{d} N(0, 1) \ , \quad (3.7)$$

where

$$\sigma^2 = \sum_i \sum_j \sum_k p_{ijk}\phi_{ijk}^2 \ . \quad (3.8)$$

Of course, (3.7) still holds if σ is replaced by its maximum likelihood estimator $\hat{\sigma}$ (obtained by substituting $\{\hat{p}_{ijk}\}$ for $\{p_{ijk}\}$ in (3.6) and (3.8)), as would be the usual case in practice. Thus, an asymptotic $100(1 - \alpha)$ percent confidence interval for $\bar{\tau}_b$ is given by

$$\bar{t}_b \pm U_{\alpha/2}\hat{\sigma}/\sqrt{n} \ ,$$

where $U_{\alpha/2}$ is the $100(1 - \alpha/2)$th percentile of the standard normal distribution. Similarly, the null hypothesis that $\bar{\tau}_b = 0$ can be tested for large n using the fact that $\sqrt{n}\bar{t}_b/\hat{\sigma}$ is asymptotically distributed $N(0, 1)$ under the null hypothesis.

To the extent that measurement of the variables is refined and there is no interaction, these procedures could be considered in practice to approximate corresponding inference procedures for the underlying measure $\tau_u(X, Y | Z)$. Notice also that $\bar{t}_b$ and its asymptotic sampling distribution may be used even if the control is multivariate or only nominal in scale, although in this last case the prior discussion on refining measurement of the control variable might not apply (e.g., controlling for sex).

We next consider an example of the calculation of the value of $\bar{t}_b$. Table 3 is taken from an article by Smith [15]. The relationship between "attitude towards abortion" and "years of schooling" is exhibited for three regions of values of the control variable "ideal number of children," for a sample of size $n = 1425$. The three corresponding sample t_b values are .2645, .0855, and .1725. These values show evidence of some interaction, though this may be due to the crudeness and variability in size of the control categories (see the end of Section 4).

2. Probabilities for Trivariate Normal Distribution with $\rho_{XY} = .64$ and $\rho_{XZ} = \rho_{YZ} = .80$ Categorized in a $2 \times 2 \times 10$ Table with $\{p_{i\cdot\cdot} = .5\}$, $\{p_{\cdot j\cdot} = .5\}$, $\{p_{\cdot\cdot k} = .1\}$, and Corresponding Weights for $\bar{\tau}_b$ and $\bar{\gamma}$[a]

				Weight	
k	i	p_{i1k}	p_{i2k}	$\bar{\tau}_b$	$\bar{\gamma}$
1	1	.0968	.0016	.011	.001
	2	.0016	.0000		
2	1	.0837	.0078	.055	.022
	2	.0078	.0007		
3	1	.0666	.0149	.107	.087
	2	.0149	.0037		
4	1	.0477	.0218	.151	.166
	2	.0218	.0088		
5	1	.0331	.0237	.175	.223
	2	.0237	.0195		

[a] The probabilities $\{p_{ijk}\}$ were obtained using the formulas given by Steck [18]. For k = 6, 7, 8, 9, 10, the weights are symmetric to k′ = 11 − k.

3. *Attitudes Toward Abortion by Years of Schooling Completed with Ideal Number of Children Controlled* [a]

Attitude toward abortion	Ideal number of children											
	None to two				Three				Four or more			
	Years of schooling				Years of schooling				Years of schooling			
	0–11	12	13+	Total	0–11	12	13+	Total	0–11	12	13+	Total
Generally disapprove	58	55	2	115	44	48	8	100	107	48	6	161
Middle position	43	51	10	104	19	37	6	62	39	38	5	82
Generally approve	104	244	104	452	67	114	24	205	66	68	10	144
Total	205	350	116	671	130	199	38	367	212	154	21	387

[a] Data from the 1972 General Social Survey of the National Data Program.

We use the sample proportions $\{\hat{p}_{ijk}\}$ as estimates of the $\{p_{ijk}\}$, and functions of the $\{\hat{p}_{ijk}\}$ as estimates of the corresponding functions of the $\{p_{ijk}\}$. For example, Δ_{11} may be estimated by

$$\hat{\Delta}_{11} = (671/1425)^2 - (115^2 + 104^2 + 452^2)/1425^2 = .1093 \; ,$$

and Δ_{21} may be estimated by

$$\hat{\Delta}_{21} = (671/1425)^2 - (205^2 + 350^2 + 116^2)/1425^2 = .1341 \; .$$

In a similar manner,

$$\hat{\Delta}_{12} = .0388 \; , \quad \hat{\Delta}_{22} = .0378 \; , \quad \hat{\Delta}_{13} = .0475 \; , \quad \hat{\Delta}_{23} = .0397.$$

Hence, $\sum (\hat{\Delta}_{1h}\hat{\Delta}_{2h})^{\frac{1}{2}} = .2028$, and the weights for the three sample t_b values are

$$(\hat{\Delta}_{11}\hat{\Delta}_{21})^{\frac{1}{2}}/\sum (\hat{\Delta}_{1h}\hat{\Delta}_{2h})^{\frac{1}{2}} = .5969, .1889, \text{and } .2142 \; .$$

Finally,

$$\bar{t}_b = .5969(.2645) + .1889(.0855) + .2142(.1725) = .211 \, ,$$

according to the sample analog of (3.5).

Alternatively, $\bar{t}_b$ could be calculated directly using the sample analog of the second formula in (3.5). The proportions of pairs of observations which are concordant and which are discordant and in the first subtable are

$$\hat{P}_1{}^{(s)} = 2[58(51 + 10 + 244 + 104) + 55(10 + 104) + 43(244 + 104) + 51(104)]/1425^2 = .0495 \; ,$$

and

$$\hat{P}_1{}^{(d)} = 2[2(43 + 51 + 104 + 244) + 55(43 + 104) + 10(104 + 244) + 51(104)]/1425^2 = .0175 \; .$$

Similarly,

$$\hat{P}_2{}^{(s)} = .0127 \; , \quad \hat{P}_2{}^{(d)} = .0094 \; , \quad \text{and} \quad \hat{P}_3{}^{(s)} = .0168 \; , \quad \hat{P}_3{}^{(d)} = .0093 \; .$$

Thus,

$$\hat{P}_s - \hat{P}_d = (.0495 - .0175) + (.0127 - .0094) + (.0168 - .0093) = .0428 \; ,$$

and since $\sum (\hat{\Delta}_{1k}\hat{\Delta}_{2k})^{\frac{1}{2}} = .2028$,

$$\bar{t}_b = .0428/.2028 = .211 \; .$$

Finally, we illustrate the calculation of the maximum likelihood estimate of the asymptotic variance. Let $i = j = k = 1$. Then

$$\hat{p}_{111} = 58/1425 = .0407 \; ,$$
$$\hat{P}_{11.1}{}^{(s)} = (51 + 10 + 244 + 104)/1425 = .2870 \; ,$$
$$\hat{P}_{11.1}{}^{(d)} = 0 \; ,$$

so that

$$\hat{\phi}_{111} = \frac{2(.2870)}{.2028} - \frac{.0428\{.1341[(671/1425) - (115/1425)] + .1093[(671/1425) - (205/1425)]\}}{[(.1093)(.1341)]^{\frac{1}{2}}(.2028)^2} = 2.0741 \; .$$

When this is calculated for all 27 combinations of (i, j, k), we obtain

$$\hat{\sigma}^2 = \sum_i \sum_j \sum_k \hat{p}_{ijk}\hat{\phi}_{ijk}{}^2 = .0407(2.0741)^2 + \ldots = .7763 \; ,$$

and hence $\hat{\sigma} = .8811$. If these data were obtained by a random sample, an approximate 95 percent confidence interval for $\bar{\tau}_b$ would be

$$.211 \pm 1.96(.8811)/(1425)^{\frac{1}{2}} = .211 \pm .046 = (.165, .257) \; .$$

The calculation of the large sample standard error of $\bar{t}_b$ can be quite cumbersome when the table dimensions are large. If the expression for ϕ_{ijk} is substituted into (3.8) and the formula is expanded, there is little cancellation and the resulting expression is messy. In practice, $\hat{\sigma}^2$ is easily obtained on a computer using the sample analog of (3.8). When $c = 1$ so that $\bar{\tau}_b$ is just Kendall's τ_b for a two-way classification, (3.8) reduces to the asymptotic variance of the random sample version of Kendall's τ_b (see [1]).

4. CATEGORY CHOICE FOR THE CONTROL VARIABLE

The choice of the number of categories for the control variable seems to be too often neglected in multivariate categorical analysis. The social science literature reveals that many researchers exert control by simply dichotomizing (see, e.g., [6, p. 150]), and that most textbooks on statistical methods for the social sciences fail to even mention the undesirability of crude categorization and collapsing of tables (see, e.g., [3, 10]).

However, unless the conditional relationship between X and Y is very similar to the unconditional relationship, $\tau(X, Y|Z)$ may not be properly revealed if c is small or if one of the levels has a much higher probability of occurrence than the others, as shown by Tables 2 and 6 and some examples in [13]. In fact, for many systems of variables it would be reasonable to assume that $\tau(X, Y|C)$ is a monotone function of C, in the sense that

$$|\tau(X, Y|C) - \tau_u(X, Y|z)| \leq |\tau(X, Y|C') - \tau_u(X, Y|z)|$$

if $z \in C$, which is itself contained in C'. In fairness, crude categorizations are probably used often in practice to make it easier to absorb the results and to ensure enough observations in each control category to have reasonable power in detecting interaction.

Many researchers who recognize the benefit of having a large c to reduce bias of the nature just described warn that the sampling variance of a partial measure increases as c increases, since the number of pairs of observations which are tied with respect to the classification of Z, and are thus used in forming the measure, decreases (see, e.g., [12, pp. 389, 393]). To see that the asymptotic variance need not increase, note Table 4, which compiles the asymptotic variance of $\sqrt{n}\bar{t}_b$ under full random sampling for various categorizations of Z, for the example of the underlying trivariate normal distribution.

4. Asymptotic Variance of $n(\bar{t}_b - \bar{\tau}_b)$, for Various $2 \times 2 \times C$ Tables with an Underlying Trivariate Normal Distribution [a]

c	$P(Z \in C_{ct})$	Variance
2	(.5,.5)	1.108
3	(.3,.4,.3)	1.216
4	(.2,.3,.3,.2)	1.207
5	(.2,.2,.2,.2,.2)	1.164
7	(.1,.1,.2,.2,.2,.1,.1)	1.229
10	(.1 each)	1.171

[a] $\rho_{XY} = .64$ and $\rho_{XZ} = \rho_{YZ} = .80$.

As another, more analytically feasible example, consider the situation in which the control variable is naturally split into the c strata $\{C_{ck}, 1 \leq k \leq c\}$, and a random sample of fixed size n_k is taken independently from the kth stratum ($\sum_k n_k = n$). Suppose that we use the partial measure $\bar{t}_b^{(s)} = \sum_k r_k t_b(\{A_{ai}\},\{B_{bj}\}|C_{ck})$, where t_b is the sample value of τ_b and $\{r_k\}$ are sample weights converging with probability one to some weights $\{\rho_k\}$ as $\{n_k \rightarrow \infty\}$, and suppose that $n_k/n \rightarrow \lambda_k$ as $n \rightarrow \infty$, for $1 \leq k \leq c$. Now

$$\frac{\sqrt{n_k}[t_b(\{A_{ai}\}, \{B_{bj}\}|C_{ck}) - \tau_b(\{A_{ai}\}, \{B_{bj}\}|C_{ck})]}{\sigma(\{A_{ai}\}, \{B_{bj}\}|C_{ck})} \xrightarrow{d} N(0, 1) \ , \quad 1 \leq k \leq c \ , \quad (4.1)$$

where $\sigma^2(\{A_{ai}\}, \{B_{bj}\}|C_{ck})$ is (3.8) applied for $c = 1$ (bivariate τ_b) to the proportions $\{p_{ijk}' = p_{ijk}/p_{..k}\}$. Since the $\{t_b(\{A_{ai}\}, \{B_{bj}\}|C_{ck}), 1 \leq k \leq c\}$ are independent, $\bar{t}_b^{(s)}$ is asymptotically normal with asymptotic mean $\bar{\tau}_b^{(s)} = \sum_k \rho_k\tau_b(\{A_{ai}\},\{B_{bj}\}|C_{ck})$ and asymptotic variance

$$\sigma_{\bar{t}_b}^2(s) = \sum_k \rho_k^2\sigma^2(\{A_{ai}\}, \{B_{bj}\}|C_{ck})/n_k \cong [\sum_k \rho_k^2\sigma^2(\{A_{ai}\}, \{B_{bj}\}|C_{ck})/\lambda_k]/n \quad (4.2)$$

That is,

$$\sqrt{n}(\bar{t}_b^{(s)} - \bar{\tau}_b^{(s)})/[\sum_k \rho_k^2\sigma^2(\{A_{ai}\}, \{B_{bj}\}|C_{ck})/\lambda_k]^{\frac{1}{2}} \xrightarrow{d} N(0, 1) \ . \quad (4.3)$$

We shall consider the simplest special case in which the sampling is proportional $\{\lambda_k = p_{..k}\}$ and $\{\rho_k = p_{..k}\}$, so that the asymptotic variance of $\sqrt{n}\bar{t}_b^{(s)}$ reduces to $\sum_k p_{..k}\sigma^2(\{A_{ai}\}, \{B_{bj}\}|C_{ck})$. Now suppose that the underlying trivariate distribution is such that

$$\lim_{C \downarrow z} \sigma^2(\{A_{ai}\}, \{B_{bj}\}|C) = \sigma^2(\{A_{ai}\}, \{B_{bj}\}|z) \quad (4.4)$$

for all z, where $\sigma^2(\{A_{ai}\}, \{B_{bj}\}|z)$ is the asymptotic variance corresponding to $t_b(\{A_{ai}\}, \{B_{bj}\}|z)$ calculated from a random sample taken at $Z = z$. Then, it can be shown that as the strata $\{C_{ck}, 1 \leq k \leq c\}$ get finer,

$$\sum_k p_{..k}\sigma^2(\{A_{ai}\}, \{B_{bj}\}|C_{ck}) \xrightarrow[c \to \infty]{} \int \sigma^2(\{A_{ai}\}, \{B_{bj}\}|z)dP_Z(z) \ , \quad (4.5)$$

where P_Z is the probability measure corresponding to the (ungrouped) distribution of Z.

When $a = b = 2$, $t_b^2(\{A_{2i}\}, \{B_{2j}\}|\cdot) = \chi^2/m$ for a sample of size m, where χ^2 is the usual statistic with df $= 1$ for the test of independence. Thus in the spurious normal case, for which $\tau_b(\{A_{2i}\}, \{B_{2j}\}|z) = 0$, and $\rho_{XY \cdot Z} = 0$ implies independence between X and Y given $Z = z$,

$$mEt_b^2(\{A_{2i}\}, \{B_{2j}\}|z) = E\chi^2 \cong 1 \text{ asymptotically (as } m \rightarrow \infty) \ ,$$

so that $\sigma^2(\{A_{2i}\}, \{B_{2j}\}|z) = 1$ for all z, and hence

$$\sum_k p_{..k}\sigma^2(\{A_{2i}\} \ , \ \{B_{2j}\}|C_{ck}) \xrightarrow[c \to \infty]{} 1 \ . \quad (4.6)$$

Thus, it appears that with stratified random sampling, the asymptotic variance of $\sqrt{n}\bar{t}_b^{(s)}$ (as $n \rightarrow \infty$) need not increase as the control variable is more finely measured (as long as $\{n_k\}$ are kept large enough to apply asymptotically derived formulas). Suppose now that, in addition, measurement of X and Y is refined in such a way that

$$\sigma^2(\{A_{ai}\} \;,\; \{B_{bj}\}\,|\,z) \xrightarrow[\substack{a\to\infty \\ b\to\infty}]{} \sigma_u{}^2(X, Y|z) \;, \qquad (4.7)$$

the asymptotic variance of the random sample value of Kendall's $\tau_u(X, Y|z)$ for observations with $Z = z$. Then one would naturally expect the asymptotic variance to decrease, since more information on the strength of the association is provided when there are fewer ties on the (X, Y) cross classification (see [1]). Under an appropriate refining process on the measurement of the three variables, it again follows that

$$\sum_k p_{\cdot\cdot k}\sigma^2(\{A_{ai}\}, \{B_{bj}\}\,|\,C_{ck}) \xrightarrow[\substack{a\to\infty \\ b\to\infty \\ c\to\infty}]{} \int \sigma_u{}^2(X, Y|z)dP_Z(z) \;. \qquad (4.8)$$

In the trivariate normal case, the conditional jointly continuous distribution of (X, Y) is identical, apart from location, at each fixed value of $Z = z$, and it follows from [8, p. 126] that this limiting value of the asymptotic variance of $\sqrt{n}\bar{t}_b{}^{(s)}$ is

$$\int_{-\infty}^{\infty} \sigma_u{}^2(X, Y|z)dP_Z(z) = 4[\tfrac{1}{9} - ((2/\pi)\sin^{-1}(\rho_{XY\cdot Z}{}^u/2))^2] \;. \qquad (4.9)$$

In particular, a maximum value of 4/9 occurs for the spurious example. By comparison, the asymptotic variance of $\sqrt{n}t_{xy}{}^u$ is $4[\frac{1}{9} - ((2/\pi)\sin^{-1}(\rho_{XY}{}^u/2))^2]$, so that the asymptotic variance of $\sqrt{n}\bar{t}_b{}^{(s)}$ exceeds this unconditional variance ($c = 1$) as measurement of the control variable is refined, roughly speaking, only when $|\rho_{XY\cdot Z}{}^u| < |\rho_{XY}{}^u|$.

Similar conclusions probably apply to other weighting schemes (such as the one used for $\bar{t}_b$ in Section 3), although the limiting variances would have different expressions and would not be as easy to compare in this context of stratified random sampling. Naturally, one should ideally choose enough observations in each stratum such that asymptotic variance formulas and asymtotic normality apply. The actual value, however, of the asymptotic sampling variance will probably not fluctuate drastically in most situations as $\{C_{ck}\}$ is changed. As long as these asymptotic considerations are satisfied, one should choose c large to reduce the difference between the measure and $\tau_u(X, Y|Z)$. If $\{P(Z \in C_{ck}),\ 1 \le k \le c\}$ are approximately equal, it appears that $c = 5$ levels would usually be sufficiently large (see Table 1 and [13, p. 407]). A guideline that has been suggested [12] for applying the asymptotic formulas requires that both the total number of conditional concordant pairs and conditional discordant pairs exceed 200.

If there is interaction, it is still advantageous to choose several control levels to try to illuminate the nature of this interaction and to make a valid comparison between each conditional association and the original unconditional association. In many practical situations, of course, restrictions in sample size necessarily place limits on such a strategy, since then a different parameter is being estimated in each level. In forming subtables to obtain a weighted average measure such as $\bar{t}_b$, the researcher is probably more likely to notice when there is interaction or nonmonotonicity than in computing a measure such as $t_{XY\cdot Z}$ (based just on bivariate measure values). When the $\{t_b(\{A_{ai}\}, \{B_{bj}\}\,|\,C_{ck})\}$ are based on independent samples from a given classification $\{C_{ck}\}$, the null hypothesis that

$$\tau_b(\{A_{ai}\}, \{B_{bj}\}\,|\,C_{c1}) = \ldots = \tau_b(\{A_{ai}\}, \{B_{bj}\}\,|\,C_{cc})$$

can be tested using the analog for τ_b of the chi-square test that Goodman and Kruskal [5, p. 318] present for testing equality of values of the measure lambda.

In many situations the control levels as naturally chosen will not be uniform in size. In that case, the separate $t_b(\{A_{ai}\}, \{B_{bj}\}\,|\,C_{ck})$ measures might indicate the existence of interaction, when in fact it would not exist under finer measurement (or vice versa). For example, suppose that $c = 3$ and that $P(Z \in C_{31}) = .1$, $P(Z \in C_{32}) = .7$, and $P(Z \in C_{33}) = .2$. Then, even if the underlying partial association were the noninteractive spurious one considered in Section 2, the three control level τ_b's as measured here would be $-.010$, .256, and .020 (see Table 5 for values of τ_b for a wide variety of control level widths). Similarly, if a finer categorization had been used for the control variable in Table 3, the observed t_b values might have been more similar, apart from sampling error. The first category is quite large, and if it had been replaced (for example) by the categories $\{(0), (1-2)\}$ or $\{(0), (1), (2)\}$, different results could have occurred. This again magnifies the importance of measuring the variables as finely as possible.

5. Value of τ_b for Relationship Between X and Y (Each Dichotomized at the Median) for Various Control Categories, for an Underlying Trivariate Normal Distribution [a]

Control quantiles	Width w of control category									
	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
	Value of τ_b									
(0,0 + w)	−.010	.020	.064	.088	.144	.201	.252	.315	.380	.443
(.1,.1 + w)	−.003	.031	.045	.096	.149	.195	.256	.319		
(.2,.2 + w)	.016	.011	.056	.101	.140	.196				
(.3,.3 + w)	−.026	.023	.062	.091						
(.4,.4 + w)	.034	.052								

[a] $\rho_{XY} = .64$ and $\rho_{XZ} = \rho_{YZ} = .80$.

APPENDIX

The derivation of the asymptotic behavior of the sample version of $\bar{\tau}_b$ follows the standard methods used by Goodman and Kruskal [5] for simple bivariate measures. Suppose that we have a random sample of size n over the full multinomial collection of probabilities $\{p_{ijk}\}$, and let $\{\hat{p}_{ijk}\}$ denote the maximum likleihood estimates of the proportions in the threefold table. Let $\hat{t}_b$, $\hat{P}_c$, $\hat{P}_d$, $\{\hat{\Delta}_{1k}\}$, and $\{\hat{\Delta}_{2k}\}$ denote the corresponding maximum likelihood estimates of τ_b, P_c, P_d, $\{\Delta_{1k}\}$, and $\{\Delta_{2k}\}$ as defined in Section 3, and assume that $\Delta_{1k} > 0$ and $\Delta_{2k} > 0$ for at least one value of k. Then

$$\sqrt{n}(\hat{t}_b - \tau_b)$$
$$= \sqrt{n}[(\hat{P}_s - \hat{P}_d)/\sum_k (\hat{\Delta}_{1k}\hat{\Delta}_{2k})^{\frac{1}{2}} - (P_s - P_d)/\sum_k (\Delta_{1k}\Delta_{2k})^{\frac{1}{2}}]$$
$$= \frac{\sqrt{n}[(\hat{P}_s - \hat{P}_d)\sum_k (\Delta_{1k}\Delta_{2k})^{\frac{1}{2}} - (P_s - P_d)\sum_k (\hat{\Delta}_{1k}\hat{\Delta}_{2k})^{\frac{1}{2}}]}{[\sum_k (\hat{\Delta}_{1k}\hat{\Delta}_{2k})^{\frac{1}{2}}][\sum_k (\Delta_{1k}\Delta_{2k})^{\frac{1}{2}}]}, \quad (A.1)$$

which has asymptotically the same distribution as

$$\frac{\sqrt{n}[(\hat{P}_s - \hat{P}_d)\sum_k (\Delta_{1k}\Delta_{2k})^{\frac{1}{2}} - (P_s - P_d)\sum_k (\hat{\Delta}_{1k}\hat{\Delta}_{2k})^{\frac{1}{2}}]}{[\sum_k (\Delta_{1k}\Delta_{2k})^{\frac{1}{2}}]^2} \quad (A.2)$$

by Slutzky's Theorem. This quantity (which we shall denote by $\sqrt{n}Q(\{\hat{p}_{ijk}\})$, being a continuous function, with continuous first partial derivatives, of asymptotically normally distributed sample proportions, is itself asymptotically normally distributed with mean 0 and variance $\sigma^2 = \mathbf{d}'\mathbf{\Sigma}\mathbf{d}$. $\mathbf{\Sigma}$ is the covariance matrix of the $\{\sqrt{n}\hat{p}_{ijk}\}$, where the covariance between $\sqrt{n}\hat{p}_{ijk}$ and $\sqrt{n}\hat{p}_{i'j'k'}$ is $\delta_{ijk,i'j'k'}p_{ijk} - p_{ijk}p_{i'j'k'}$, with $\delta_{ijk,i'j'k'} = 1$ if $i = i'$, $j = j'$, and $k = k'$, and 0 otherwise, and $\mathbf{d}$ is the vector of partial derivatives of Q with respect to the $\{\hat{p}_{ijk}\}$ evaluated at $\{p_{ijk}\}$. Now

$$\hat{P}_s = 2\sum_i \sum_j \sum_k \hat{p}_{ijk}(\sum_{i'>i}\sum_{j'>j} \hat{p}_{i'j'k}) ,$$

so that

$$(\partial\hat{P}_s/\partial\hat{p}_{ijk})\,|_{\{p_{ijk}\}}$$
$$= 2(\sum_{i'>i}\sum_{j'>j} p_{i'j'k} + \sum_{i'<i}\sum_{j'<j} p_{i'j'k}) = 2P_{ij\cdot k}{}^{(s)} . \quad (A.3)$$

Similarly,

$$(\partial\hat{P}_d/\partial\hat{p}_{ijk})\,|_{\{p_{ijk}\}} = 2(\sum_{i'>i}\sum_{j'<j} p_{i'j'k} + \sum_{i'<i}\sum_{j'>j} p_{i'j'k}) = 2P_{ij\cdot k}{}^{(d)}$$

In addition,

$$(\partial[\sum_h (\hat{\Delta}_{1h}\hat{\Delta}_{2h})^{\frac{1}{2}}]/\partial\hat{p}_{ijk})\,|_{\{p_{ijk}\}}$$
$$= \frac{\Delta_{2k}(p_{\cdot\cdot k} - p_{i\cdot k}) + \Delta_{1k}(p_{\cdot\cdot k} - p_{\cdot jk})}{(\Delta_{1k}\Delta_{2k})^{\frac{1}{2}}}, \quad (A.4)$$

so that

$$(\partial Q/\partial\hat{p}_{ijk})\,|_{\{p_{ijk}\}} = \frac{2(P_{ij\cdot k}{}^{(s)} - P_{ij\cdot k}{}^{(d)})}{\sum_h (\Delta_{1h}\Delta_{2h})^{\frac{1}{2}}}$$
$$- \frac{(P_s - P_d)[\Delta_{2k}(p_{\cdot\cdot k} - p_{i\cdot k}) + \Delta_{1k}(p_{\cdot\cdot k} - p_{\cdot jk})]}{[\sum_h (\Delta_{1h}\Delta_{2h})^{\frac{1}{2}}]^2(\Delta_{1k}\Delta_{2k})^{\frac{1}{2}}}. \quad (A.5)$$

Thus, letting $\phi_{ijk} = (\partial Q/\partial\hat{p}_{ijk})\,|_{\{p_{ijk}\}}$, the asymptotic variance of $\sqrt{n}\hat{t}_b$ is

$$\sigma^2 = \sum_i \sum_j \sum_k \sum_{i'} \sum_{j'} \sum_{k'} \phi_{ijk}\phi_{i'j'k'}(\delta_{ijk,i'j'k'}p_{ijk} - p_{ijk}p_{i'j'k'})$$
$$= \sum_i \sum_j \sum_k p_{ijk}\phi_{ijk}^2 - (\sum_i \sum_j \sum_k p_{ijk}\phi_{ijk})^2 . \quad (A.6)$$

Using the fact that $P_k{}^{(s)} = \sum_i \sum_j p_{ijk}P_{ij\cdot k}{}^{(s)}$ and $P_s = \sum_k P_k{}^{(s)}$ (and similarly for P_d), it can be seen through substitution that $\sum_i \sum_j \sum_k p_{ijk}\phi_{ijk} = 0$. Thus

$$\sigma^2 = \sum_i \sum_j \sum_k p_{ijk}\phi_{ijk}^2 . \quad (A.7)$$

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