EXAM - STA 6329 - Practice Problem Solutions

(1) $A = \begin{bmatrix} T & U \\ V & W \end{bmatrix}$ $\quad B = \begin{bmatrix} T^{-1}+T^{-1}UQ^{-1}VT^{-1} & -T^{-1}UQ^{-1} \\ -Q^{-1}VT^{-1} & Q^{-1} \end{bmatrix}$

$Q = W - VT^{-1}U$

$$(AB)_{11} = \underbrace{TT^{-1}}_{I} + \underbrace{TT^{-1}}_{I}UQ^{-1}VT^{-1} - UQ^{-1}VT^{-1} = I + UQ^{-1}VT^{-1} - UQ^{-1}VT^{-1} = I$$

$$(AB)_{12} = -\underbrace{TT^{-1}}_{I}UQ^{-1} + UQ^{-1} = -UQ^{-1} + UQ^{-1} = 0$$

$$(AB)_{21} = VT^{-1} + VT^{-1}UQ^{-1}VT^{-1} - WQ^{-1}VT^{-1} = VT^{-1} - \underbrace{\overbrace{(W - VT^{-1}U)}^{Q}Q^{-1}}_{I}VT^{-1}$$

$$= VT^{-1} - VT^{-1} = 0$$

$$(AB)_{22} = -VT^{-1}UQ^{-1} + WQ^{-1} = (W - VT^{-1}U)Q^{-1} = QQ^{-1} = I$$

(2) $A = \begin{bmatrix} 1 & 2 & 7 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ 0 & F_{22} & F_{23} & F_{24} \\ 0 & 0 & F_{33} & F_{34} \\ 0 & 0 & 0 & F_{44} \end{bmatrix}$ $\quad AF = I$

$$\boxed{F_{ii} = A^{-1}_{ii} = 1 \qquad i = 1, 2, 3, 4}$$

$F_{44} = 1 \quad A[3,] \times F[,4] = 0 = F_{34} + 3F_{44} = F_{34} + 3 \Rightarrow \boxed{F_{34} = -3}$

$A[2,] \times F[,4] = 0 = F_{24} - 2F_{34} = F_{24} - 2(-3) = F_{24} + 6 \Rightarrow \boxed{F_{24} = -6}$

$A[1,] \times F[,4] = 0 = F_{14} + 2F_{24} + 7F_{34} + 4F_{44} = F_{14} + 2(-6) + 7(-3) + 4(1)$

$\Rightarrow F_{14} - 12 - 21 + 4 = 0 \Rightarrow \boxed{F_{14} = 29}$

$F_{33} = 1 \quad A[2,] \times F[,3] = 0 = F_{23} - 2F_{33} \Rightarrow \boxed{F_{23} = 2}$

$A[1,] \times F[,3] = 0 = F_{13} + 2F_{23} + 7F_{33} = F_{13} + 4 + 7 \Rightarrow \boxed{F_{13} = -11}$

2\. continued

(2)

$$F_{22} = 1 \qquad A[1,]\times F[,2] = 0 = F_{12} + 2F_{22} \Rightarrow \boxed{F_{12} = -2}$$

Check
$$AF = \begin{bmatrix} 1 & 2 & 7 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -11 & 29 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark$$

3\.
$$P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad P' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\Rightarrow PP' = \begin{bmatrix} \cos^2\theta + \sin^2\theta = 1 & -\cos\theta\sin\theta + \sin\theta\cos\theta = 0 \\ -\sin\theta\cos\theta + \cos\theta\sin\theta = 0 & \sin^2\theta + \cos^2\theta = 1 \end{bmatrix} \checkmark$$

4\.
$$A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 2 & 3 \end{bmatrix}$$
a) $\operatorname{rank}(A) = 2$ rows 1, 2 lin. indep. / cols " " "

b)
$$\begin{bmatrix} 3 & -1 & 2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \\ R_{31} & R_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 3R_{11} - R_{21} + 2R_{31} = 1 \quad (ia)$$
$$-R_{11} + 2R_{21} + 3R_{31} = 0 \quad (iia)$$

$R_{21} = \frac{1}{5}(1 - 11R_{31})$

$R_{11} = \frac{1}{5}(2 - 7R_{31})$

$\cancel{(ia) + 3(iia) \Rightarrow 5R_{21} + 11R_{31} = 1 \Rightarrow R_{21} = \frac{1}{5}(1 - 11R_{31})}$ ✓

$\cancel{2(ia) + (iia) \Rightarrow 5R_{11} + 7R_{31} = 2 \Rightarrow R_{11} = \frac{1}{5}(2 - 7R_{31})}$ ✓

$\cancel{\Rightarrow -\frac{1}{5}(2 - 7R_{31}) + 2\left(\frac{1}{5}\right)(1 - 11R_{31}) + 3R_{31} = 0}$ →

$\cancel{\Rightarrow \left(-\frac{2}{5} + \frac{2}{5}\right) + R_{31}\left[\frac{7}{5} - \frac{22}{5} + \frac{15}{5}\right] = 0 = 0 + 0R_{31}}$

~~(need something else)~~

4. Continued

Let $R_{31} = 1 \Rightarrow R_{21} = \frac{1}{5}\left(1 - 11(1)\right) = -2$, $R_{11} = \frac{1}{5}\left(2 - 7(1)\right) = -1$

$3(-1) - (-2) + 2 = 1$, $\quad -(-1) + 2(-2) + 3(1) = 0$

$$
\begin{aligned}
3R_{12} - R_{22} + 2R_{32} &= 0 \qquad \text{(ib)} \\
-R_{12} + 2R_{22} + 3R_{32} &= 1 \qquad \text{(iib)}
\end{aligned}
$$

$0 + 5R_{22} + 11R_{32} = 3 \qquad$ (ib) + 3(iib) $\Rightarrow R_{22} = \frac{1}{5}(3 - 11R_{32})$

$5R_{12} + 0 + 7R_{32} = 1 \qquad$ 2(ib) + (iib) $\Rightarrow R_{12} = \frac{1}{5}(1 - 7R_{32})$

Let $R_{32} = 3 \Rightarrow R_{22} = -6$, $R_{12} = -4$

$3(-4) - (-6) + 2(3) = 0$, $\quad -(-4) + 2(-6) + 3(3) = 1$

$$\Rightarrow R = \begin{bmatrix} -1 & -4 \\ -2 & -6 \\ 1 & 3 \end{bmatrix}$$

5. $\quad A = \begin{bmatrix} T & 0 \\ V & W \end{bmatrix} \qquad G = \begin{bmatrix} T^- & 0 \\ -W^-VT^- & W^- \end{bmatrix}$

$$AGA = \begin{bmatrix} TT^- & 0 \\ VT^- - WW^-VT^- & WW^- \end{bmatrix} \begin{bmatrix} T & 0 \\ V & W \end{bmatrix}$$

$$= \begin{bmatrix} TT^-T & 0 \\ VT^-T - WW^-VT^-T + WW^-V & WW^-W \end{bmatrix} = \begin{bmatrix} T & 0 \\ V^* & W \end{bmatrix}$$

$\mathcal{Q}(V) \subset \mathcal{Q}(T) \Rightarrow V = AT$ for some A

$$
\begin{aligned}
\Rightarrow (AGA)_{21} &= ATT^-T - WW^-ATT^-T + WW^-AT \\
&= AT - WW^-AT + WW^-AT = AT = V
\end{aligned}
$$

(6) $A = \begin{bmatrix} 6 & -3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ $\quad G \equiv 3\times 2 \quad A_{11} = \begin{bmatrix} 6 & -3 \\ 0 & 1 \end{bmatrix}$

$$A_{11}^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix} \Rightarrow G = \frac{1}{6}\begin{bmatrix} 1 & 3 \\ 0 & 6 \\ 0 & 0 \end{bmatrix}$$

$$AGA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & -3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = A$$

(7) $\underset{m\times n}{A}$ w/ $\text{rank}(A) = r \Rightarrow A = \underset{m\times m}{B}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\underset{n\times n}{K}$ $\quad B, K \equiv$ nonsingular

$G = K^{-1}\begin{bmatrix} I_r & U \\ V & W \end{bmatrix} B^{-1}$ $\equiv$ g-inverse of A for any U, V, W of correct dims

$$AGA = B\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} K K^{-1}\begin{bmatrix} I_r & U \\ V & W \end{bmatrix} B^{-1} B \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} K$$

$$= B\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_r & U \\ V & W \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} K = B\begin{bmatrix} I_r & U \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} K$$

$$= B\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} K = A$$

(8.) $\underset{n\times n}{A^2 = A}$ $(\neq 0)$ w/ $\text{rank}(A) = r$ $\qquad A = \underset{n\times r}{B}\,\underset{r\times n}{L}$ w/ $\text{rank}(B) = \text{rank}(L) = r$

$$A = A^2 = BLBL = BL \Rightarrow LB = I_r \Rightarrow tr(LB) = r$$

$$\Rightarrow tr(BL) = tr(LB) = r \Rightarrow tr(A) = r$$

(9) $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow X'X = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} \Rightarrow (X'X)^{-1} = \frac{1}{15-9}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \Rightarrow P = X(X'X)^{-1}X' = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$

$PP = \frac{1}{36}\begin{bmatrix} 30 & 12 & -6 \\ 12 & 12 & 12 \\ -6 & 12 & 30 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$ ✓

(10) $\underset{n\times n}{A} \equiv$ idempotent $\quad \underset{n\times n}{B} \equiv$ nonsingular $\quad$ show $B^{-1}AB \equiv$ idempotent

$B^{-1}AB\,B^{-1}AB = B^{-1}AIAB = B^{-1}AAB = B^{-1}AB$

(11) $A'A \equiv$ idempotent $\Rightarrow A'A = A'AA'A$

Cor. 5.3.3.: $AB = AC$ iff $A'AB = A'AC$

Let $B = I \quad C = A'A \Rightarrow AI = AA'A = A$

$\Rightarrow AA' = AA'AA' \Rightarrow AA' \equiv$ idempotent

(12) $X_0 \equiv$ particular solution to $AX = B$ $\quad Z^* \equiv$ some solution to $AZ = 0$

Let $X^* = X_0 + Z^* \Rightarrow AX^* = A(X_0 + Z^*) = AX_0 + AZ^* = B + 0 = B$

$\Rightarrow X^* \equiv$ solution to $AX = B$

(13) $\underset{\sim}{x}^* = A^-\underset{\sim}{b} + (I - A^-A)\underset{\sim}{y}$ $\quad$ for some $\underset{\sim}{y}$ (a solution to $A\underset{\sim}{x} = \underset{\sim}{b}$ is $A^-\underset{\sim}{b}$)

$A\underset{\sim}{x}^* = AA^-\underset{\sim}{b} + (A - AA^-A)\underset{\sim}{y} = AA^-\underset{\sim}{b} + (A-A)\underset{\sim}{y} = AA^-\underset{\sim}{b} = \underset{\sim}{b}$

(since $A^-b \equiv$ solution to $A\underset{\sim}{x} = \underset{\sim}{b}$)

(14) $A = \begin{bmatrix} 10 & 5 & 5 \\ 5 & 5 & 0 \\ 5 & 0 & 5 \end{bmatrix}$ $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 70 \\ 30 \\ 40 \end{bmatrix}$

a) G_1 : $A[1:2, 1:2]^{-1} = \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix}^{-1} = \frac{1}{50-25}\begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$\Rightarrow G_1 = \frac{1}{5}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$G_2 : A[2:3, 2:3]^{-1} = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\underset{\sim}{x}^* = \frac{1}{5}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 70 \\ 30 \\ 40 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 70-30 \\ -70+60 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix}$

$\underset{\sim}{x}^{**} = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 70 \\ 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 8 \end{bmatrix}$

(15) $\underset{n \times n}{A}$ $\mathcal{N}(A) = \{\underset{\sim}{x} : A\underset{\sim}{x}\} = 0$ $\mathcal{C}(I-A) \equiv$ set $(I-A)F$ for all F.

$(\underset{\sim}{x} \neq 0)$ $\underset{\sim}{x} \in \mathcal{N}(A) \Rightarrow A\underset{\sim}{x} = 0$ $x \in \mathcal{C}(I-A) = \underset{\sim}{x} = (I-A)\underset{\sim}{f}$ for some $\underset{\sim}{f} \neq \underset{\sim}{0}$

$A\underset{\sim}{x} = 0 \Rightarrow A(I-A)\underset{\sim}{f} = 0 = (A - AA)\underset{\sim}{f} = \underset{\sim}{0} \Rightarrow A - AA = 0 \Rightarrow A = AA$

$\mathcal{N}(A) = \mathcal{C}(I-A) \Rightarrow \forall\, \underset{\sim}{x}$ $(I-A)\underset{\sim}{x} \in \mathcal{N}(A) \Rightarrow A(I-A)\underset{\sim}{x} = 0$

Let $\underset{\sim}{x} = \underset{\sim}{u}_i$ $i = 1, \ldots, n$ (columns of I_n)

$\Rightarrow A(I-A) = 0 \Rightarrow A - A^2 = 0 \Rightarrow A = A^2$

16 $A = \begin{bmatrix} 10 & 5 & 5 \\ 5 & 5 & 0 \\ 5 & 0 & 5 \end{bmatrix}$ $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 70 \\ 30 \\ 40 \end{bmatrix}$

$K' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ a) $\mathcal{R}(K') \subset \mathcal{R}(A) \ni \exists\ T$ s.t. $K' = TA$

$T = \begin{bmatrix} \frac{1}{5} & 0 & -\frac{1}{5} \\ 0 & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$

b) $K' \underset{\sim}{x}^{*} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ $K' \underset{\sim}{x}^{**} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

c) $A' Y = K \Rightarrow Y = (A')^{-} K = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$

$K' \underset{\sim}{x}^{*} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$ $Y^{*\prime} \underset{\sim}{b} = \frac{1}{5}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 70 \\ 30 \\ 40 \end{bmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$