Section 13.1 - Definitions, Notation, Special Cases

13.1.a. Negative and Positive Pairs

$A = \{a_{ij}\} \equiv n \times n$ matrix   2 elements in different rows/columns: $a_{ij}, a_{i'j'}$

Pair $\{a_{ij}, a_{i'j'}\} \equiv \begin{cases} \underline{\text{negative}} & \text{if one is above and to right of other } (i'>i, j'<j) \text{ or } (i'<i, j'>j) \\ \underline{\text{positive}} & \text{if one is above and to left of other } (i'>i, j'>j) \text{ or } (i'<i, j'<j) \end{cases}$

	$i'>i$	$i'<i$
$j'>j$	+	−
$j'<j$	−	+

Example ($n \geq 4$)

$\{a_{34}, a_{22}\} \equiv$ positive $\{a_{34}, a_{41}\} \equiv$ negative

- Consider n elements of A, no two of which in same row or column

$i_1, j_1; \ldots; i_n, j_n$ elements $(i_1, \ldots, i_n), (j_1, \ldots, j_n) \equiv$ permutations of $1, \ldots, n$

# of pairs than can be formed from $1, \ldots, n \equiv \binom{n}{2}$

$\sigma_n(i_1, j_1; \ldots; i_n, j_n) \equiv$ # of the $\binom{n}{2}$ pairs that are negative

Properties of $\sigma_n(i_1, j_1; \ldots, i_n, j_n)$

1) Value of σ_n not affected by permuting its n pairs of arguments (can sort on row number or column number).

$\sigma_3(2,3;\ 1,2;\ 3,1) = \sigma_3(1,2;\ 2,3;\ 3,1) = \sigma_3(3,1;\ 1,2;\ 2,3)$

2) $\sigma_n(i_1, j_1; \ldots; i_n, j_n) = \sigma_n(j_1, i_1; \ldots, j_n, i_n)$ $\sigma_3(2,3;\ 1,2;\ 3,1) = \sigma_3(3,2;\ 2,1;\ 1,3)$

- For any sequence of n distinct integers: $i_1, \ldots, i_n$ define:

$\phi_n(i_1, \ldots, i_n) = p_1 + \ldots + p_{n-1}$ where $p_k \equiv$ # of integers in $i_{k+1}, \ldots, i_n < i_k$ $k=1,\ldots,n-1$

Example $\phi_5(3, 7, 2, 1, 4) = 2 + 3 + 1 + 0 = 6$

- For any permutation $i_1, \ldots, i_n$ of $1, 2, \ldots, n$:

$\sigma_n(1, i_1; \ldots; n, i_n) = \sigma_n(i_1, 1; \ldots, i_n, n) = \phi_n(i_1, \ldots, i_n)$

13.1.6. Definition of a Determinant

The determinant of $\underset{n\times n}{A} = \{a_{ij}\}$ denoted by $|A|$ or $\det A$ or $\det(A)$

is defined by $|A| = \sum (-1)^{\sigma_n(1,j_1;\ldots;n,j_n)} a_{1j_1} \cdots a_{nj_n}$ (1.2a)

or equivalently by $|A| = \sum (-1)^{\phi_n(j_1,\ldots,j_n)} a_{1j_1} \cdots a_{nj_n}$ (1.2b)

where $j_1,\ldots,j_n$ = permutation of $1,\ldots,n$ and sum is over all permutations.

Determinant obtained in principle by:

1) Form all possible products, each of n factors, by choosing one element from each row and column of A
2) In each product, count number of negative pairs among the $\binom{n}{2}$ pairs of elements that can be generated from the n elements forming the particular product. If number of negative pairs is even +, if odd − times product.
3) Sum all signed products

Examples

1×1 matrix: $A = (a_{11})$ $\quad |A| = a_{11}$

2×2 matrix: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = (-1)^0 a_{11}a_{22} + (-1)^1 a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21}$

3×3 matrix: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad |A| = (-1)^0 a_{11}a_{22}a_{33} + (-1)^1 a_{11}a_{23}a_{32} + (-1)^1 a_{12}a_{21}a_{33} + (-1)^2 a_{12}a_{23}a_{31} + (-1)^2 a_{13}a_{21}a_{32} + (-1)^3 a_{13}a_{22}a_{31}$

$\Rightarrow |A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$

− For partitioned matrix: $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$: $\left|\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right|$ written as $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$

− Alternative definition:

$$|A| = \sum (-1)^{\sigma_n(i_1,1;\ldots;i_n,n)} a_{i_11} \cdots a_{i_nn} \qquad (1.6a)$$

$$= (-1)^{\phi_n(i_1,\ldots,i_n)} a_{i_11} \cdots a_{i_nn} \qquad (1.6b)$$

Definitions (1.2) and (1.6) are equivalent (just ordered differently)

$a_{1j_1} \cdots a_{nj_n} = a_{i_1 1} \cdots a_{i_n n}$ $\quad i_1, \ldots, i_n \equiv$ permutation of $1, \ldots, n$ s.t. $j_{i_1} = 1, \ldots, j_{i_n} = n$

$\sigma_n(1, i_1; \ldots, n, i_n) = \sigma_n(i_1, j_{i_1}; \ldots; i_n, j_{i_n}) = \sigma(i_1, 1; \ldots, i_n, n)$

$\Rightarrow (-1)^{\sigma_n(1, j_1; \ldots, n, j_n)} a_{1j_1} \cdots a_{nj_n} = (-1)^{\sigma_n(i_1, 1; \ldots; i_n, n)} a_{i_1 1} \cdots a_{i_n n}$

$\Rightarrow$ (1.2) and (1.6) are elementwise and sum equals.

Number of terms in each summation is $n!$

13.1.c. Diagonal, Triangular, Permutation Matrices

Lemma 13.1.1. $\underset{n \times n}{A} \equiv$ upper or lower triangular, then $|A| = a_{11} a_{22} \cdots a_{nn}$

Proof: Lower triangular $A \equiv \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ only nonzero term in summation is $j_1 = 1, \ldots, j_n = n$ w/ $\phi^n(1, 2, \ldots, n) = 0$

Formally: $j_1, \ldots, j_n \equiv$ arbitrary permutation of $1, \ldots, n$ and suppose $a_{1j_1} \cdots a_{nj_n} \neq 0$

$(a_{ij_i} \neq 0 \quad i = 1, \ldots, n) \Rightarrow j_1 = 1 \quad (a_{1j} = 0 \; j > 1)$ If $j_1 = 1$, then $j_2 = 2$ $(a_{2j} = 0 \; j > 2)$

If $j_1 = 1, j_2 = 2, \ldots, j_{i-1} = i-1$, then $j_i = i$ and so forth by induction. $\square$

Corollary 13.1.2. – The determinant of a diagonal matrix equals product of its diagonal elements. Special Cases: $|0| = 0 \quad |I| = 1$

Lemma 13.1.3. For $n \times n$ permutation P w/ $1^{st} \ldots n^{th}$ rows or columns being rows or columns $k_1, \ldots, k_n$ of I_n, then $|P| = (-1)^{\phi_n(k_1, \ldots, k_n)}$

Proof: Letting $A = P$ sum (1.6) has all 0 terms except when $i_1 = k_1, \ldots, i_n = k_n$ (1) and its multiplier is $(-1)^{\phi_n(k_1, \ldots, k_n)}$

Section 13.2 - Some Basic Properties of Matrices

Lemma 13.2.1. For any $\underset{n\times n}{A}$, $|A'| = |A|$

Proof: $A \equiv \{a_{ij}\}$ $A' = \{b_{ij}\}$ $i,j = 1,\dots,n$ $(b_{ij} = a_{ji})$

$$|A'| = \sum (-1)^{\phi_n(j_1,\dots,j_n)} b_{1j_1}\cdots b_{nj_n} = \sum (-1)^{\phi_n(j_1,\dots,j_n)} a_{j_1 1}\cdots a_{j_n n} = |A|$$

Lemma 13.2.2. $\underset{n\times n}{A} = [\underset{\sim}{a}_1 \dots \underset{\sim}{a}_i \dots \underset{\sim}{a}_n]$ $\underset{n\times n}{B} = [\underset{\sim}{a}_1 \cdots k\underset{\sim}{a}_i \cdots \underset{\sim}{a}_n]$

$|B| = k|A|$ (Same holds if single row multiplied by k)

Proof: $|B| = \sum (-1)^{\phi_n(j_1,\dots,j_n)} a_{j_1 1}\cdots k a_{j_i i}\cdots a_{j_n n} = k \sum (-1)^{\phi_n(j_1,\dots,j_n)} a_{j_1 1}\cdots a_{j_i i}\cdots a_{j_n n} = k|A|$ □

Corollary 13.2.3. If one or more rows (or columns) of $\underset{n\times n}{A}$ are null, $|A| = 0$

Proof: Suppose row i of $A \equiv$ null and let $B \equiv$ matrix that keeps A, but multiplies all elements of row i of A by $k=0$. Then $|A| = |B| = 0|A| = 0$ □

Corollary 13.2.4. For any $\underset{n\times n}{A}$, scalar k, $|kA| = k^n|A|$

Proof: $|B| = \sum (-1)^{\phi_n(j_1,\dots,j_n)} k a_{j_1 1}\cdots k a_{j_n n} = k^n \sum (-1)^{\phi_n(j_1,\dots,j_n)} a_{j_1 1}\cdots a_{j_n n} = k^n |A|$ □

Corollary 13.2.5. For any $\underset{n\times n}{A}$: $|-A| = (-1)^n |A|$ $(k=-1)$

Theorem 13.2.6. $\underset{n\times n}{B}$ formed from $\underset{n\times n}{A}$ by interchanging 2 rows or columns $\Rightarrow$ $|B| = -|A|$

Proof: Let B interchange rows $i, i+1$ in A:

$$\Rightarrow |B| = \sum (-1)^{\phi_n(j_1,\dots,j_n)} b_{1j_1}\cdots b_{nj_n} = \sum (-1)^{\phi_n(j_1,\dots,j_n)} a_{1j_1}\cdots a_{i-1,j_{i-1}} a_{i+1,j_i} a_{i,j_{i+1}} a_{i+2,j_{i+2}} \cdots a_{nj_n}$$

$$= (-1)\sum (-1)^{\phi_n(j_1,\dots,j_n)} a_{1j_1}\cdots a_{i-1,j_{i-1}} a_{i,j_{i+1}} a_{i+1,j_i} a_{i+2,j_{i+2}}\cdots a_{nj_n} = -|A|$$

(since $\phi_n(j_1,\dots,j_{i-1},j_{i+1},j_i,j_{i+2},\dots,j_n) = \begin{cases} \phi_n(j_1,\dots,j_n)+1 & \text{if } j_{i+1} > j_i \\ \phi_n(j_1,\dots,j_n)-1 & \text{if } j_{i+1} < j_i \end{cases}$

extends to not necessarily neighboring rows □

Theorem 13.2.7. $\underset{n\times p}{B}$ w/ $p<n$ $\underset{n\times q}{C}$ w/ $q=n-p$ then :

$$|B\ \ C| = (-1)^{pq}\,|C\ \ B| \qquad \underset{p\times n,\ q\times n}{B\ \ C} \Rightarrow \left|\begin{matrix}B\\C\end{matrix}\right| = (-1)^{pq}\left|\begin{matrix}C\\B\end{matrix}\right|$$

Proof: $[C\ \ B] = [\underset{\sim}{c}_1 \cdots \underset{\sim}{c}_q\ \underset{\sim}{b}_1 \cdots \underset{\sim}{b}_p] \to [\underset{\sim}{b}_1 \cdots \underset{\sim}{b}_p\ \underset{\sim}{c}_1 \cdots \underset{\sim}{c}_q] = (B\ \ C]$

(pq total interchanges) $\Rightarrow |B\ C| = (-1)^{pq}|C\ \ B|$ $\quad\circ$

Lemma 13.2.8. If two row or two columns of $\underset{n\times n}{A}$ are identical, then $|A|=0$

Proof: Suppose rows i, k of A are identical Let $B = A$ w/ row i, k interchanged

$\Rightarrow B = A \Rightarrow |B| = |A|$, but by Th. 13.2.6. $|B| = -|A| \Rightarrow |A| = |B| = -|A| \Rightarrow |A| = 0$

LEMMA 13.2.9. If a row or column of $\underset{n\times n}{A}$ is scalar multiple of another row/ column then $|A|=0$. Proof: $A = \begin{bmatrix}\underset{\sim}{a}_1'\\ \vdots\\ \underset{\sim}{a}_n'\end{bmatrix}$ suppose $\underset{\sim}{a}_s' = k\underset{\sim}{a}_i'$ $s\neq i$ $B = \begin{bmatrix}\underset{\sim}{a}_1'\\ \vdots\\ k\underset{\sim}{a}_i'\\ \vdots\\ \underset{\sim}{a}_n'\end{bmatrix}$

Then (Lem. 13.2.?, 13.2.8.) $\Rightarrow |B| = k|A| = 0$

Theorem 13.2.10. Let $\underset{n\times n}{B} \equiv$ matrix formed from $\underset{n\times n}{A}$ by adding to any row/ col of A, scalar multiples of one or more other rows/colums Then: $|B| = |A|$

Proof: Let $\underset{\sim}{a}_i' = [a_{i1} \cdots a_{in}]$ and $\underset{\sim}{b}_i' = [b_{i1} \cdots b_{in}] \equiv i^{th}$ rows of A, B $\quad i=1,\dots,n$

For some integer $s \in \{1,\dots,n\}$ and scalars $k_1,\dots,k_{s-1}, k_{s+1},\dots k_n$

Let $\underset{\sim}{b}_s = \underset{\sim}{a}_s' + \sum_{\substack{i=1\\ i\neq s}}^{n} k_i \underset{\sim}{a}_i'$ and $\underset{\sim}{b}_i' = \underset{\sim}{a}_i'$ $i \neq s$

$$\Rightarrow |B| = \sum (-1)^{\phi_n(j_1,\dots,j_n)} b_{1j_1}\cdots b_{nj_n} = \sum (-1)^{\phi_n(j_1,\dots,j_n)} a_{1j_1}\cdots a_{s-1,j_{s-1}}\Big(a_{sj_s} + \sum_{i\neq s} k_i a_{ij_s}\Big) a_{s+1,j_{s+1}}\cdots a_{nj_n}$$

$$= |A| + \sum_{i\neq s}\sum (-1)^{\phi_n(j_1,\dots,j_n)} a_{1j_1}\cdots a_{s-1,j_{s-1}}\,k_i a_{ij_s}\, a_{s+1,j_{s+1}}\cdots a_{nj_n} = |A| + \sum_{i\neq s}|B_i|$$

$B_i \equiv$ matrix formed from A ~~w/ row~~ replacing row s of A w/ $k_i \underset{\sim}{a}_i'$

and $j_1,\dots,j_n \equiv$ permutations of $1,\dots,n$ and summations over all permutations.

By Lem. 13.2.9. : $|B_i| = 0$ $(i\neq s) \Rightarrow |B| = |A|$ $\quad\square$

Theorem 13.2.11. For any $\underset{n\times n}{A}$ and unit upper/lower triangular $\underset{n\times n}{T}$:

$$|AT| = |TA| = |A|$$

Proof: Consider A post-multiplied by lower triangular T

$T_i \equiv$ matrix formed by replacing from I_n, the i^{th} column of T $(i=1,\dots,n)$

$I_n = [\underset{\sim}{u}_1 \cdots \underset{\sim}{u}_n]$ $\quad T_1 = [\underset{\sim}{t}_1\ \underset{\sim}{u}_2 \cdots \underset{\sim}{u}_n] \cdots\ T_n = [\underset{\sim}{u}_1 \cdots \underset{\sim}{u}_{n-1}\ \underset{\sim}{t}_n]$

Note: $T_1T_2 = [\underset{\sim}{t}_1\ \underset{\sim}{t}_2\ \underset{\sim}{u}_3 \dots \underset{\sim}{u}_n] \Rightarrow T_1T_2\cdots T_n = T \Rightarrow AT = AT_1T_2\cdots T_n$

Let $B_0 = A$ $\quad B_i := AT_1\dots T_i \quad i=1,\dots,n-1$ $\quad$ To show $|AT|=|T|$ suffices to show $|B_{i-1}|=|B_{i-1}T|$

~~Column i of~~ $\cancel{B_{i-1}T_i}$ $\quad A = \begin{bmatrix} a_1' \\ \vdots \\ a_n' \end{bmatrix}$

$B_0 = A$ $\quad \cancel{B_1 = AT_1 = [\underset{\sim}{a}_1\underset{\sim}{t}_1\ \underset{\sim}{a}_2 \cdots \underset{\sim}{a}_n]} \quad \cancel{B_2 = AT_1T_2 = [\underset{\sim}{a}_1\underset{\sim}{t}_1}$

$B_1 = AT_1 = [A\underset{\sim}{t}_1\ A \dots A]$ $\qquad B_2 = AT_1T_2 = [A\underset{\sim}{t}_1\ A\underset{\sim}{t}_2\ A \dots A]$

Columns of $B_{i-1}T_i$, B_{i-1} are the same except column i

$B_{i-1} = [A\underset{\sim}{t}_1 \dots A\underset{\sim}{t}_{i-1}\ A \cdots A]$ $\qquad B_{i-1}T_i = [A\underset{\sim}{t}_1 \cdots A\underset{\sim}{t}_{i-1}\ A\underset{\sim}{t}_i\ A \dots A]$

$\Rightarrow$ i^{th} column of $B_{i-1}t_i$ is i^{th} column of B_{i-1} plus scalar multiples of columns $i+1,\dots,n$ of B_{i-1}

$\Rightarrow$ (By Th. 13.2.10.) $|B_{i-1}T_i| = |B_{i-1}| \Rightarrow |AT| = |A|$

(Holds for pre- and post-multiplying A by unit upper/lower triangular matrices) $\square$

Note: $n=3$ $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad T = \begin{bmatrix} 1 & 0 & 0 \\ t_{21} & 1 & 0 \\ t_{31} & t_{32} & 1 \end{bmatrix}$ $\quad T_1 = \begin{bmatrix} 1 & 0 & 0 \\ t_{21} & 1 & 0 \\ t_{31} & 0 & 1 \end{bmatrix}$ $T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_{32} & 1 \end{bmatrix}$

$T_3 = I$ $\quad B_1 = AT_1 = \begin{bmatrix} a_{11}+a_{12}t_{21}+a_{13}t_{31} & a_{12} & a_{13} \\ a_{21}+a_{22}t_{21}+a_{23}t_{31} & a_{22} & a_{23} \\ a_{31}+a_{32}t_{21}+a_{33}t_{31} & a_{32} & a_{33} \end{bmatrix} = [\underset{\sim}{b}_{11}\ \underset{\sim}{b}_{12}\ \underset{\sim}{b}_{13}]$

$B_2 = AT_1T_2 = \begin{bmatrix} \underset{\sim}{b}_{11} & \begin{matrix} a_{12}+a_{13}t_{32} \\ a_{22}+a_{23}t_{32} \\ a_{32}+a_{33}t_{33} \end{matrix} & \begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix} \end{bmatrix} = [\underset{\sim}{b}_{11}\ \ \underset{\sim}{b}_{12} + t_{32}\underset{\sim}{b}_{13}\ \ \underset{\sim}{b}_{13}] = [\underset{\sim}{b}_{21}\ \underset{\sim}{b}_{22}\ \underset{\sim}{b}_{23}]$

$\Rightarrow B_3 = AT_1T_2T_3 = B_2$ (Note $B_1 = [\underset{\sim}{b}_{01} + t_{21}\underset{\sim}{b}_{02} + t_{31}\underset{\sim}{b}_{03}\ \ \underset{\sim}{b}_{02}\ \ \underset{\sim}{b}_{03}]$

Section 13.3 – Partitioned Matrices, Products, and Inverses

Theorem 13.3.1. $\underset{m\times m}{T}$ $\underset{n\times m}{V}$ $\underset{n\times n}{W}$ Then: $\begin{vmatrix} T & 0 \\ V & W \end{vmatrix} = \begin{vmatrix} W & V \\ 0 & T \end{vmatrix} = |T||W|$

Proof: $A = \begin{pmatrix} T & 0 \\ V & W \end{pmatrix} = \{a_{ij}\}$ $\quad i, j = 1, \dots, m+n$

$$|A| = \sum (-1)^{\phi_{m+n}(j_1, \dots, j_{m+n})} a_{1j_1} \cdots a_{m+n, j_{m+n}} \qquad j_1, \dots, j_{m+n} \equiv \text{permutation of } 1, \dots, m+n$$

Non-zero terms are when $j_1, \dots, j_m \equiv$ permutation of $1, \dots, m$ $\quad j_{m+1}, \dots, j_{m+n} \equiv$ perm of $m+1, \dots, m+n$

$$\begin{bmatrix} a_{1,m+1} & \cdots & a_{1,m+n} \\ \vdots & & \vdots \\ a_{m,m+1} & \cdots & a_{m,m+n} \end{bmatrix} = 0$$

For any such permutation:

$$a_{1j_1} \cdots a_{m+n, j_{m+n}} = t_{1j_1} \cdots t_{mj_m} w_{1, j_{m+1}-m} \cdots w_{n, j_{m+n}-m}$$
$$= t_{1j_1} \cdots t_{mj_m} w_{1k_1} \cdots w_{nk_n} \qquad k_i = j_{m+i} - m \quad i = 1, \dots, n$$

Also $\phi_{m+n}(j_1, \dots, j_{m+n}) = \phi_m(j_1, \dots, j_m) + \phi_n(j_{m+1}, \dots, j_{m+n})$

$$= \phi_m(j_1, \dots, j_m) + \phi_n(j_{m+1} - m, \dots, j_{m+n} - m) = \phi_m(j_1, \dots, j_m) + \phi_n(k_1, \dots, k_n)$$

$$= \sum_{\substack{t \\ \text{perms}}} \sum_{\substack{w \\ \text{perms}}} (-1)^{\phi_m(j_1, \dots, j_m) + \phi_n(k_1, \dots, k_n)} t_{1j_1} \cdots t_{mj_m} w_{1k_1} \cdots w_{nk_n}$$

$$= \left[\sum_{\substack{t \\ \text{perms}}} (-1)^{\phi_m(j_1, \dots, j_m)} t_{1j_1} \cdots t_{mj_m} \right] \left[\sum_{\substack{w \\ \text{perms}}} (-1)^{\phi_n(k_1, \dots, k_n)} w_{1k_1} \cdots w_{nk_n} \right] = |T||W| \quad \square$$

– Repeated use of Th. 13.3.1 $\rightarrow$ Results for upper and lower Block Triangular matrices

$$\begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ 0 & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{vmatrix} = |A_{11}||A_{22}| |A_{rr}| \qquad \begin{vmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ B_{r1} & B_{r2} & \cdots & B_{rr} \end{vmatrix} = |B_{11}||B_{22}| \cdots |B_{rr}|$$

and Block Diagonal matrices:

$$\begin{vmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & A_{rr} \end{vmatrix} = |A_{11}||A_{22}| \cdots |A_{rr}|$$

Corollary 13.3.2. $\underset{m\times m}{T}, \underset{n\times m}{V}, \underset{n\times n}{W} \Rightarrow \begin{vmatrix} 0 & T \\ W & V \end{vmatrix} = \begin{vmatrix} V & W \\ T & 0 \end{vmatrix} = (-1)^{mn}|T||W|$

(Th. 13.2.7, Th. 13.3.1.)

Corollary 13.3.3. $\underset{n\times n}{W}, \underset{n\times n}{V} \Rightarrow \begin{vmatrix} 0 & -I_n \\ W & V \end{vmatrix} = \begin{vmatrix} V & W \\ -I_n & 0 \end{vmatrix} = |W|$

(Makes use of previous result w/ $T = -I_n$, n even $\Rightarrow n^2$ even, n odd $\Rightarrow n^2$ odd)

Proof: $(-1)^{m}|-I_n||W| = (-1)^{n^2}(-1)^{n}|W| = (-1)^{n(n+1)}|W| = |W|$ since $n(n+1)$ = even $\square$

- Making use of Th. 13.3.1, Cor 13.3.3, Th. 13.2.11 for $\underset{n\times n}{A}, \underset{n\times n}{B}$

$$|A||B| = \begin{vmatrix} A & 0 \\ -I & B \end{vmatrix} = \left| \begin{pmatrix} A & 0 \\ -I & B \end{pmatrix} \underbrace{\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}}_{\text{Unit upper triangular}} \right| = \begin{vmatrix} A & AB \\ -I & 0 \end{vmatrix} = |AB|$$

Theorem 13.3.4. $\underset{n\times n}{A}\ \underset{n\times n}{B} \Rightarrow |AB| = |A||B|$

- special cases: 1) $|A_1 A_2 \cdots A_k| = |A_1||A_2|\cdots|A_k|$ for $A_1, \dots, A_k \equiv n\times n$

2) $A \equiv n\times n \Rightarrow |A^k| = |A|^k$

Corollary 13.3.5. $\underset{n\times n}{A} \Rightarrow |A'A| = |A'||A| = |A||A| = |A|^2$ (Lem. 13.2.1)

Corollary 13.3.6. For any orthogonal matrix P, $|P| = \pm 1$

Proof: $|P|^2 = |P'P| = |I| = 1 \Rightarrow |P| = \pm 1$ $\square$

Theorem 13.3.7. $\underset{n\times n}{A}$ $A \equiv$ nonsingular (invertible) iff $|A| \neq 0$ in which case:

$$|A^{-1}| = \frac{1}{|A|}$$

Proof: Show: 1) If $A \equiv$ nonsingular, then $|A| \neq 0$ and $|A^{-1}| = \frac{1}{|A|}$

2) If $A \equiv$ singular, then $|A| = 0$

Suppose $A \equiv$ nonsingular: $|A^{-1}||A| = |A^{-1}A| = |I| = 1 \Rightarrow |A| \neq 0$, $|A^{-1}| = \frac{1}{|A|}$

Suppose $A \equiv$ singular $\Rightarrow A\underset{\sim}{c} = 0$ for some $\underset{\sim}{c} \neq 0$ (Linearly dependent columns)

e.g. say column $s \equiv$ linear combination of others: $\underset{\sim}{a}_s = \sum_{i\neq s} k_i \underset{\sim}{a}_i$ for scalars $k_1, \dots, k_{s-1}, k_{s+1}, \dots, k_n$

Let $B \equiv A$ w/ adding $-\sum_{i\neq s} k_i \underset{\sim}{a}_i + \underset{\sim}{a}_s \Rightarrow \underset{\sim}{b}_s = \underset{\sim}{0} \Rightarrow |B| = |A| = 0$ (Th. 13.2.10, Cor. 13.2.3) $\square$

Theorem 13.3.8 $\underset{m\times m}{T}$ $\underset{m\times n}{U}$ $\underset{n\times m}{V}$ $\underset{n\times n}{W}$ If T is nonsingular,

then: $\begin{vmatrix} T & U \\ V & W \end{vmatrix} = \begin{vmatrix} W & V \\ U & T \end{vmatrix} = |T|\,|W - VT^{-1}U|$

Proof $T \equiv$ nonsingular $\Rightarrow \begin{pmatrix} I & 0 \\ VT^{-1} & W - VT^{-1}U \end{pmatrix} \begin{bmatrix} T & U \\ 0 & I \end{bmatrix} = \begin{bmatrix} T & U \\ V & W \end{bmatrix}$

Th. 13.3.4.: $|AB| = |A||B| \Rightarrow \begin{vmatrix} T & U \\ V & W \end{vmatrix} = \begin{vmatrix} I & 0 \\ VT^{-1} & W - VT^{-1}U \end{vmatrix} \begin{vmatrix} T & U \\ 0 & I \end{vmatrix}$

Th. 13.3.1.: $\begin{vmatrix} T & U \\ 0 & I \end{vmatrix} = |T||I| = |T|(1) = |T|$

$\begin{vmatrix} I & 0 \\ VT^{-1} & W - VT^{-1}U \end{vmatrix} = |I|\,|W - VT^{-1}U| = (1)|W - VT^{-1}U| = |W - VT^{-1}U|$

$\Rightarrow \begin{vmatrix} T & U \\ V & W \end{vmatrix} = |T||W - VT^{-1}U|$ similar for $\begin{pmatrix} W & V \\ U & T \end{pmatrix}$ D

Section 13.4 – Computational Approach

If $\underset{n\times n}{A}$ can be decomposed into product of simpler matrices (wrt computing $|\cdot|$)

Then $|A| = |A_1 \cdots A_k| = |A_1| \cdots |A_k|$ can be easier to compute.

- QR Decomposition: $\underset{n\times n}{A} = QR$ $Q \equiv$ orthogonal, $R \equiv$ triangular

$\Rightarrow |A| = \pm |R|$ since $|Q| = \pm 1$

$\Rightarrow |A| = \pm \prod_{i=1}^{n} r_{ii}$ $\left(R = \{r_{ij}\}\right.$ and $\left.|R| = \prod_{i=1}^{n} r_{ii}\right)$

Section 13.5. – Cofactors

Let $\underset{n\times n}{A} = \{a_{ij}\}$ $A_{ij} \equiv A$ with row i, column j removed ("struck out")

$|A_{ij}| \equiv$ minor of element a_{ij}

"signed" minor: $(-1)^{i+j}|A_{ij}| \equiv$ cofactor of a_{ij}

Determinant of A can be expanded in terms of cofactors of the n elements of any row or column of A

Theorem 13.5.1. Let $\underset{n\times n}{A} \equiv \{a_{ij}\}$ $\quad \alpha_{ij} \equiv$ cofactor of a_{ij}

Then for $i=1,\dots,n$: $|A| = \sum_{j=1}^{n} a_{ij}\alpha_{ij} = a_{i1}\alpha_{i1} + \dots + a_{in}\alpha_{in}$

" $j=1,\dots,n$ $|A| = \sum_{i=1}^{n} a_{ij}\alpha_{ij} = a_{1j}\alpha_{1j} + \dots + a_{nj}\alpha_{nj}$

Proof - See Harville (very long)

Theorem 13.5.2. $\underset{n\times n}{A} \equiv \{a_{ij}\}$ $\quad \alpha_{ij} \equiv$ cofactor of a_{ij}

Then for $i' \neq i = 1,\dots,n$: $\sum_{j=1}^{n} a_{ij}\alpha_{i'j} = a_{i1}\alpha_{i'1} + \dots + a_{in}\alpha_{i'n} = 0$ (5.3)

" $j' \neq j = 1,\dots,n$ $\sum_{i=1}^{n} a_{ij}\alpha_{ij'} = a_{1j}\alpha_{1j'} + \dots + a_{nj}\alpha_{nj'} = 0$ (5.4)

Proof of (5.3): $B \equiv$ matrix A w/ row i' replaced by row i of A

$\Rightarrow |B| = 0$ $\quad$ Let $B = \{b_{kj}\}$ $k,j=1,\dots,n$ $\Rightarrow$ Cofactor of $b_{i'j}$ = Cofactor of $a_{i'j}$ $\quad j=1,\dots,n$

Then: $\sum_{j=1}^{n} a_{ij}\alpha_{i'j} = \sum_{j=1}^{n} b_{i'j}\alpha_{i'j} = |B| = 0$ $\quad \square$

$\downarrow$
Th. 13.5.1.

— For any matrix $\underset{n\times n}{A} \equiv \{a_{ij}\}$ then $n\times n$ matrix w/ i,j element $= \alpha_{ij}$ is called the matrix of cofactors (cofactor matrix) of A

Its transpose is the adjoint matrix of A.

$$\operatorname{adj} A = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}' = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{n1} \\ \vdots & & \vdots \\ \alpha_{1n} & & \alpha_{nn} \end{bmatrix}$$ "adjoint of A"

$A \equiv$ symmetric $\Rightarrow$ matrix of cofactors $\equiv$ symmetric $= \operatorname{adj}(A)$

Theorem 13.5.3. For $\underset{n\times n}{A}$ $\quad A\operatorname{adj}(A) = (\operatorname{adj} A)A = |A| I_n$

Proof: $A = \{a_{ij}\}$ $\quad \alpha_{ij} \equiv$ Cofactor of a_{ij} $\quad i,j = 1,\dots,n$

i,i element of $A\operatorname{adj}(A) = a_{i1}\alpha_{i1} + \dots + a_{in}\alpha_{in} = |A|$ $\quad i=1,\dots,n$

$i \neq i'$: i,i' element " " $= a_{i1}\alpha_{i'1} + \dots + a_{in}\alpha_{i'n} = 0$ $\quad i \neq i' = 1,\dots,n$

$$\Rightarrow A\operatorname{adj}(A) = \begin{bmatrix} |A| & & 0 \\ & \ddots & \\ 0 & & |A| \end{bmatrix} = |A| I \quad \square$$

Corollary 13.5.4. $\underset{n\times n}{A} \equiv$ nonsingular matrix. Then:

$$\text{adj}(A) = |A| A^{-1} \text{ or equivalently } A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

~~Proof: If $A \equiv$ invertible, then $|A^{-1}| = \frac{1}{|A|}$~~

Proof: From Th. 13.5.3. $A\,\text{adj}(A) = |A| I \Rightarrow A^{-1}A\,\text{adj}(A) = A^{-1}|A| = \text{adj}(A)$

Section 13.6. - Vandermonde Matrices

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

(polynomial regression setting)

square matrix

Let
$$T = \begin{bmatrix} 1 & -x_n & 0 & \cdots & 0 & 0 \\ 0 & 1 & -x_n & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & -x_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \Rightarrow VT = \begin{bmatrix} \underset{\sim}{1}_{n-1} & A \\ 1 & 0' \end{bmatrix}$$

w/
$$\underset{(n-1)\times(n-1)}{A} = \begin{bmatrix} x_1 - x_n & x_1(x_1 - x_n) & \cdots & x_1^{n-2}(x_1 - x_n) \\ x_2 - x_n & x_2(x_2 - x_n) & \cdots & x_2^{n-2}(x_2 - x_n) \\ \vdots & & & \\ x_{n-1} - x_n & x_{n-1}(x_{n-1} - x_n) & \cdots & x_{n-1}^{n-2}(x_{n-1} - x_n) \end{bmatrix}$$

By post-multiply V by T:
1) Column 1 stays same
2) Columns 2-n get scaler multiples of preceding column.
3) Last row is $[1, 0, \ldots 0]$

Let $\underset{(n-1)\times(n-1)}{D} = \text{diag}\{x_i - x_n\}$ $i = 1, \ldots, n-1$

$$\Rightarrow A = DW \quad \text{where } W = V[1:n-1, 1:n-1] = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n-1} & \cdots & x_{n-1}^{n-2} \end{bmatrix}$$

$\Rightarrow$ A = Product of diagonal $(n-1)\times(n-1)$ matrix and submatrix of V w/ last row and column removed. $W \equiv (n-1)\times(n-1)$ Vandermonde matrix

Note: $|V| = |VT| = \begin{vmatrix} \underset{\sim}{f}_{n-1} & A \\ 1 & 0 \end{vmatrix} = (-1)^{n-1}|A| = (-1)^{n-1}|D||W| = |-D|\,|W|$

(T = unit upper triangular) $= (x_n - x_1)\cdots(x_n - x_{n-1})\,|W|$

$\Rightarrow$ Determinant of $n \times n$ Vandermonde matrix is related to that of $(n-1)\times(n-1)$ Vandermonde matrix.

$n=2$: $|V_2| = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1$

$n=3$: $|V_3| = (x_3 - x_1)(x_3 - x_2)|V_2| = (x_3 - x_1)(x_3 - x_2)(x_3 - x_1)$

In general: $|V| = \prod_{\substack{i,j \\ j<i}}^{n} (x_i - x_j) = (x_n - x_1)\cdots(x_n - x_{n-1})(x_{n-1} - x_1)\cdots(x_{n-1} - x_{n-2})\cdots(x_2 - x_1)$

Note: $|V| \neq 0$ iff $x_j \neq x_i \;\forall\; j < i = 1, \ldots, n$ (No "repeat" x's)

$\operatorname{rank}(V) = r \equiv$ # distinct values among $x_1, \ldots, x_n$.

Let $i_1 < i_2 < \cdots < i_r$ be r integers s.t. $x_{i_1}, \ldots, x_{i_r} \equiv$ distinct

$\Rightarrow$ Rows $i_1, \ldots, i_r$ of V span $\mathcal{R}(V)$ (all other rows duplicates)

$\Rightarrow \operatorname{rank}(V) = \dim(\mathcal{R}(V)) \leq r$

$r \times r$ submatrix of V: Rows $i_1, \ldots, i_r$, Columns $1, 2, \ldots, r$

$$\begin{bmatrix} 1 & x_{i_1} & x_{i_1}^2 & \cdots & x_{i_1}^{r-1} \\ 1 & x_{i_2} & x_{i_2}^2 & & x_{i_2}^{r-1} \\ \vdots & & \vdots & & \vdots \\ 1 & x_{i_r} & x_{i_r}^2 & & x_{i_r}^{r-1} \end{bmatrix} \equiv \text{nonsingular Vandermonde matrix}$$

$\Rightarrow \operatorname{rank}(V) \geq r$ (Th. 4.4.10) and $\operatorname{rank}(V) \leq r \Rightarrow \operatorname{rank}(V) = r$

Section 13.7 – Results on Determinant of Sum of 2 Matrices

$\underset{n\times n}{A}, \underset{n\times n}{B}$ In general $|A+B| \neq |A| + |B|$

Theorem 13.7.1. $\underset{n\times n}{A}, \underset{n\times n}{B}$ then $|A+B| = \sum_{\{i_1,\dots,i_r\}} |C_k^{\{i_1,\dots,i_r\}}|$ (7.1)

where: $k \equiv$ any particular value of $1,\dots,n$ $\{i_1,\dots,i_r\} \equiv$ subset of $1,\dots,k$

and summation is over all 2^k such subsets.

and $C_k^{\{i_1,\dots,i_r\}} \equiv$ $n\times n$ matrix w/ last $n-k$ rows identical to last $n-k$ rows of $A+B$

w/ rows $i_1,\dots,i_r$ " " rows of A

w/ rows $i_{r+1},\dots,i_k$ " " " " B

Theorem 13.7.2. $\underset{n\times n}{A} = \{a_i'\}$ $\underset{n\times n}{B} = \{b_i'\}$, $C = \{c_i'\}$ $i=1,\dots,n$

If for some k, $c_k' = a_k' + b_k'$ and $c_i' = a_i' = b_i'$ $i = 1,\dots,k-1,k+1,\dots,n$

then $|C| = |A| + |B|$

Proof: let $A \equiv \{a_{kj}\}$ $B \equiv \{b_{kj}\}$ $C \equiv \{c_{kj}\}$ $k,j = 1,\dots,n$

$\alpha_{kj} \equiv$ cofactor of a_{kj} similar for β_{kj}, γ_{kj} of b_{kj}, c_{kj}, respectively

$\Rightarrow \alpha_{kj} = \beta_{kj} = \gamma_{kj}$ (only row k is different)

$\Rightarrow$ (by Th. 13.5.1.) $|C| = \sum_{j=1}^n c_{kj}\alpha_{kj} = \sum_{j=1}^n (a_{kj}+b_{kj})\alpha_{kj} = \sum_{j=1}^n a_{kj}\alpha_{kj} + \sum_{j=1}^n b_{kj}\beta_{kj}$

$= |A| + |B|$ □

Proof of Th. 13.7.1 (Induction) By Th. 13.7.2: $|A+B| = |C_1^{\{1\}}| + |C_1^{\phi}|$ $\phi \equiv$ empty set

$\Rightarrow$ (7.1) holds for $k=1$ Suppose it holds for $k = k^*-1$

$\Rightarrow |A+B| = \sum_{\{i_1,\dots,i_r\}} |C_{k^*-1}^{\{i_1,\dots,i_r\}}|$ where $\{i_1,\dots,i_r\} \equiv$ subset of $1,\dots,k^*-1$
summation over all 2^{k^*-1} subsets

Show it holds for k^*. By Th. 13.7.2. : $|C_{k^*-1}^{\{i_1,\dots,i_r\}}| = |C_{k^*}^{\{i_1,\dots,i_r,k^*\}}| + |C_{k^*}^{\{i_1,\dots,i_r\}}|$

for any subset $\{i_1,\dots,i_r\}$ of $1,\dots,k^*-1$

→

The 2^{k^*} subsets of $1, \dots, k^*$ consist of:

2^{k^*-1} subsets of $1, \dots, k^*-1$, plus the 2^{k^*-1} subsets that augment them w/ k^*
(2^{k^*-1} w/ k^*, 2^{k^*-1} w/out k^*)

$\Rightarrow |A+B| = \sum_{\{i_1, \dots, i_r\}} |C_{k^*}^{\{i_1, \dots, i_r\}}|$ where $\{i_1, \dots, i_r\} \equiv$ subset of $1, \dots, k^*$ and sum over all 2^{k^*} such subsets □

<u>Theorem 13.7.3.</u> $\underset{n\times n}{B}$, $D = \mathrm{diag}\{d_1, \dots, d_n\}$

Then: $$|D+B| = \sum_{\{i_1, \dots, i_r\}} d_{i_1} \cdots d_{i_r} |B^{\{i_1, \dots, i_r\}}| \quad (7.2)$$ where: $\{i_1, \dots, i_r\} \equiv$ subset of $1, \dots, n$, sum over all 2^n subsets

$B^{\{i_1, \dots, i_r\}} \equiv$ Principal submatrix of B by removing rows/cols $i_1, \dots, i_r$

Term in 7.2 corresponding to empty set is $|B|$ (remove no rows/cols)
" " " " " $\{1, \dots, n\}$ is $d_1 \cdots d_n = |D|$

<u>Proof: See Harville</u>

<u>Corollary 13.7.4.</u> For any $\underset{n\times n}{B}$ and any scalar x:

$$|B + xI| = \sum_{r=0}^{n} x^r \sum_{\{i_1, \dots, i_r\}} |B^{\{i_1, \dots, i_r\}}| \quad (7.3)$$

where: $\{i_1, \dots, i_r\} \equiv$ r-dimensional subset of $1, \dots, n$
and second sum is over all $\binom{n}{r}$ subsets.

$B^{\{i_1, \dots, i_r\}} \equiv (n-r) \times (n-r)$ principal submatrix of B, eliminating rows/cols $i_1, \dots, i_r$

For $r = n$: $\sum_{\{i_1, \dots, i_r\}} |B^{\{i_1, \dots, i_r\}}| \equiv 1$

Note that (7.3) $\equiv$ polynomial in x w/ coefficients:

x^0: $|B|$ $\qquad x^{n-1}$: $\mathrm{tr}(B)$ $\qquad x^n$: 1

↑ full matrix $\qquad$ ↑ only one element (for each one "left in"