Matrix Decompositions of Square Matrix A

LU Decomposition (aka LDU) – **L** ≡ unit upper triangular, **U ≡** upper triangular (unit if diagonal **D** included)

• Unique if **A** is positive definite • Exists for symmetric, nonnegative definite **A**

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \mathbf{L}=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \qquad \mathbf{U}=\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \qquad \mathbf{A}=\mathbf{LU}$$

$$\Rightarrow \quad 1\left(u_{1j}\right)=a_{1j} \quad j=1,\ldots,n \qquad l_{21}u_{1j}+u_{2j}=a_{2j} \quad j=1,\ldots,n \qquad l_{k1}u_{1j}+\ldots+l_{k,k-1}u_{k-1,j}+u_{kj}=a_{kj} \quad \begin{matrix} k=2,\ldots,n \\ j=1,\ldots,n \end{matrix}$$

QR Decomposition – **Q** ≡ orthogonal, **R** ≡ upper triangular (**A** need not be square, can be $m \times n$)

$$\underset{n\times n}{\mathbf{A}}=\left[\mathbf{a}_1 \ \cdots \ \mathbf{a}_n\right] \qquad \underset{n\times n}{\mathbf{B}}=\left[\mathbf{b}_1 \ \cdots \ \mathbf{b}_n\right] \qquad \mathbf{b}_1=\mathbf{a}_1 \qquad \mathbf{b}_2=\mathbf{a}_2-x_{12}\mathbf{b}_1 \quad x_{12}=\frac{\mathbf{a}_2\bullet\mathbf{b}_1}{\mathbf{b}_1\bullet\mathbf{b}_1}=\frac{\mathbf{a}_2{}'\mathbf{b}_1}{\mathbf{b}_1{}'\mathbf{b}_1}$$

$$\mathbf{b}_j=\mathbf{a}_j-\sum_{i=j-1}^{1}x_{ij}\mathbf{b}_i \quad j=2,\ldots,n \quad x_{ij}=\frac{\mathbf{a}_j\bullet\mathbf{b}_i}{\mathbf{b}_i\bullet\mathbf{b}_i}=\frac{\mathbf{a}_j{}'\mathbf{b}_i}{\mathbf{b}_i{}'\mathbf{b}_i} \qquad \left\|\mathbf{b}_i\right\|=\sqrt{\mathbf{b}_i{}'\mathbf{b}_i} \qquad \mathbf{X}=\begin{bmatrix} 1 & x_{12} & \cdots & x_{1n} \\ 0 & 1 & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{E}=\text{diag}\left\{\left\|\mathbf{b}_i\right\|\right\} \qquad \mathbf{D}=\text{diag}\left\{\left\|\mathbf{b}_i\right\|^{-1}\right\}=\mathbf{E}^{-1} \qquad \mathbf{Q}=\mathbf{BD} \qquad \mathbf{R}=\mathbf{EX} \qquad \mathbf{A}=\mathbf{QR}$$

Cholesky Decomposition (aka square root) **A** ≡ positive definite, **U** ≡ upper triangular

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \mathbf{U}=\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \qquad \mathbf{A}=\mathbf{U}'\mathbf{U}=\begin{bmatrix} u_{11} & 0 & \cdots & 0 \\ u_{12} & u_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

$$\Rightarrow \quad \left(u_{11}\right)^2=a_{11} \quad u_{11}u_{1j}=a_{1j} \quad \Rightarrow \quad u_{1j}=\frac{a_{1j}}{u_{11}} \quad j=1,\ldots,n$$

$$\Rightarrow \quad \left(u_{22}\right)^2=a_{22} \quad u_{12}u_{1j}+u_{22}u_{2j}=a_{2j} \quad \Rightarrow \quad u_{2j}=\frac{a_{2j}-u_{12}u_{1j}}{u_{22}} \quad j=1,\ldots,n$$

$$\Rightarrow \quad \left(u_{kk}\right)^2=a_{kk} \quad u_{1k}u_{1j}+\ldots+u_{k-1,k}u_{k-1,j}+u_{kk}u_{kj}=a_{kj} \quad \Rightarrow \quad u_{kj}=\frac{a_{kj}-\sum_{i=k-1}^{1}u_{ik}u_{ij}}{u_{kk}} \quad \begin{matrix} k=2,\ldots,n \\ j=1,\ldots,n \end{matrix}$$

Spectral Decomposition Based on Eigenvalues/Eigenvectors of a diagonalizable square matrix **A**

Consider the linear system: $\mathbf{Aw} = \lambda\mathbf{w} \quad \Rightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{0}$ with **determinantal equations**: $|\mathbf{A} - \lambda\mathbf{I}| = 0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad c_0 + c_1\lambda + \ldots + c_n\lambda^n = 0$$

with coefficients $c_0, \ldots, c_n$ depending on **A**

with **eigenvalues**: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and **eigenvectors**: $\mathbf{w}_1, \ldots, \mathbf{w}_n$ s.t. $(\mathbf{A} - \lambda_j\mathbf{I})\mathbf{w}_j = \mathbf{0}$

where (for $i \neq j$) $\mathbf{w}_i'\mathbf{w}_j = 0$ if $\lambda_i \neq \lambda_j$ rank(**A**) = # of non-zero eigenvalues

$$\text{Letting: } \mathbf{q}_j = \frac{1}{\sqrt{\mathbf{w}_j'\mathbf{w}_j}}\mathbf{w}_j \qquad \mathbf{Q} = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{bmatrix} \qquad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad \mathbf{QQ}' = \mathbf{Q}'\mathbf{Q} = \mathbf{I} \qquad \mathbf{Q\Lambda Q}' = \mathbf{A}$$

Notes: $\mathbf{A}^{-1} = \mathbf{Q\Lambda}^{-1}\mathbf{Q}'$ since $\mathbf{Q\Lambda}^{-1}\mathbf{Q}'\mathbf{Q\Lambda Q}' = \mathbf{QQ}' = \mathbf{I} \qquad \mathbf{A}^{1/2} = \mathbf{Q\Lambda}^{1/2}\mathbf{Q}'$

Singular Value Decomposition – **U**, **V** ≡ orthogonal, **D** ≡ diagonal (**A** need not be square, can be $m \times n$)

Eigenvalue Decompositions of $\mathbf{A}'\mathbf{A}$ and $\mathbf{AA}'$: $\mathbf{A}'\mathbf{A} = \mathbf{V\Lambda V}' \quad \mathbf{AA}' = \mathbf{U\Lambda U}' \quad \mathbf{U},\mathbf{V} \equiv$ orthogonal $\quad \mathbf{\Lambda} \equiv$ diagonal

$$\Rightarrow \quad \mathbf{A} = \mathbf{U\Lambda}^{1/2}\mathbf{V}' \quad \mathbf{A}'\mathbf{A} = \mathbf{V\Lambda}^{1/2}\mathbf{U}'\mathbf{U\Lambda}^{1/2}\mathbf{V}' = \mathbf{V\Lambda V}' \quad \mathbf{AA}' = \mathbf{U\Lambda}^{1/2}\mathbf{V}'\mathbf{V\Lambda}^{1/2}\mathbf{U}' = \mathbf{U\Lambda U}'$$

Data Examples:

A		
13	6	5
6	12	6
5	6	13

A	New Delhi	Seoul	Taipei	Ho Chi Minh Cit	Shanghai	Vladivostok	Jakarta	Tokyo	Manila	Singapore	Karachi	Bangkok
Temp01	57	28	61	80	40	8	80	43	79	81	66	80
Temp02	62	32	61	82	42	14	80	44	80	82	70	83
Temp03	72	42	65	84	48	27	81	49	82	83	77	85
Temp04	84	54	71	86	59	40	82	58	85	83	84	87
Temp05	91	64	77	86	68	49	82	66	85	83	87	86
Temp06	93	72	82	84	75	56	82	72	83	84	89	85
Temp07	88	77	85	83	83	64	81	78	82	83	86	85
Temp08	86	78	85	83	82	68	81	81	81	83	84	84
Temp09	84	70	81	83	75	61	82	75	82	82	85	84
Temp10	79	59	76	82	66	48	83	65	82	82	83	83
Temp11	69	45	69	81	55	30	82	56	81	81	76	81
Temp12	60	33	63	80	44	14	81	48	79	80	68	79