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- Exam 1 – Fall 2019 **PRINT** Name ANSWER KEY

...s at $\alpha = 0.05$ significance level

...ons are based on the following 2 regression models.

Model 1: $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i \quad i = 1,\ldots,n \quad \varepsilon_i \sim NID\left(0,\sigma^2\right)$ Model 2: $\mathbf{Y} = \mathbf{X\beta} + \boldsymbol{\varepsilon} \quad \mathbf{X} \equiv n\times 2 \quad \boldsymbol{\beta} \equiv 2\times 1 \quad \boldsymbol{\varepsilon} \sim N\left(\mathbf{0},\sigma^2\mathbf{I}\right)$

Given: $\dfrac{d(\mathbf{a}'\mathbf{x})}{d\mathbf{x}} = \mathbf{a} \qquad \dfrac{d(\mathbf{x}'\mathbf{A}\mathbf{x})}{d\mathbf{x}} = 2\mathbf{A}\mathbf{x}$ (A symmetric) $\qquad E(\mathbf{Y}'\mathbf{A}\mathbf{Y}) = tr(\mathbf{A}\mathbf{V}_\mathbf{Y}) + \boldsymbol{\mu}_\mathbf{Y}'\mathbf{A}\boldsymbol{\mu}_\mathbf{Y}$

Suppose **Y** is distributed as follows with nonsingular matrix **V**:

$\mathbf{Y} \sim N\left(\boldsymbol{\mu},\sigma^2\mathbf{V}\right) \qquad r(\mathbf{V}) = n \qquad$ then if **AV** is idempotent:

1. $\mathbf{Y}'\left(\dfrac{1}{\sigma^2}\mathbf{A}\right)\mathbf{Y}$ is distributed non-central χ^2 with: (a) df $= r(\mathbf{A})$ and (b) Noncentrality parameter: $\Omega = \dfrac{1}{2\sigma^2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$

2. $\mathbf{Y}'\mathbf{A}\mathbf{Y}$, $\mathbf{Y}'\mathbf{B}\mathbf{Y}$ independent if $\mathbf{AVB} = \mathbf{0}$ $\quad$ $\mathbf{Y}'\mathbf{A}\mathbf{Y}$, $\mathbf{BY}$ independent if $\mathbf{BVA} = \mathbf{0}$

Critical Values for t and F-distributions
F-distributions indexed by numerator df across top of table

df	t(.05)	t(.025)	F(.05,1)	F(.05,2)	F(.05,3)	F(.05,4)	F(.05,5)	F(.05,6)
1	6.314	12.706	161.448	199.500	215.707	224.583	230.162	233.986
2	2.920	4.303	18.513	19.000	19.164	19.247	19.296	19.330
3	2.353	3.182	10.128	9.552	9.277	9.117	9.013	8.941
4	2.132	2.776	7.709	6.944	6.591	6.388	6.256	6.163
5	2.015	2.571	6.608	5.786	5.409	5.192	5.050	4.950
6	1.943	2.447	5.987	5.143	4.757	4.534	4.387	4.284
7	1.895	2.365	5.591	4.737	4.347	4.120	3.972	3.866
8	1.860	2.306	5.318	4.459	4.066	3.838	3.687	3.581
9	1.833	2.262	5.117	4.256	3.863	3.633	3.482	3.374
10	1.812	2.228	4.965	4.103	3.708	3.478	3.326	3.217
11	1.796	2.201	4.844	3.982	3.587	3.357	3.204	3.095
12	1.782	2.179	4.747	3.885	3.490	3.259	3.106	2.996
13	1.771	2.160	4.667	3.806	3.411	3.179	3.025	2.915
14	1.761	2.145	4.600	3.739	3.344	3.112	2.958	2.848
15	1.753	2.131	4.543	3.682	3.287	3.056	2.901	2.790
16	1.746	2.120	4.494	3.634	3.239	3.007	2.852	2.741
17	1.740	2.110	4.451	3.592	3.197	2.965	2.810	2.699
18	1.734	2.101	4.414	3.555	3.160	2.928	2.773	2.661
19	1.729	2.093	4.381	3.522	3.127	2.895	2.740	2.628
20	1.725	2.086	4.351	3.493	3.098	2.866	2.711	2.599
30	1.697	2.042	4.171	3.316	2.922	2.690	2.534	2.421
40	1.684	2.021	4.085	3.232	2.839	2.606	2.449	2.336
50	1.676	2.009	4.034	3.183	2.790	2.557	2.400	2.286
60	1.671	2.000	4.001	3.150	2.758	2.525	2.368	2.254
70	1.667	1.994	3.978	3.128	2.736	2.503	2.346	2.231
80	1.664	1.990	3.960	3.111	2.719	2.486	2.329	2.214
90	1.662	1.987	3.947	3.098	2.706	2.473	2.316	2.201
100	1.660	1.984	3.936	3.087	2.696	2.463	2.305	2.191
110	1.659	1.982	3.927	3.079	2.687	2.454	2.297	2.182
120	1.658	1.980	3.920	3.072	2.680	2.447	2.290	2.175
130	1.657	1.978	3.914	3.066	2.674	2.441	2.284	2.169
140	1.656	1.977	3.909	3.061	2.669	2.436	2.279	2.164
150	1.655	1.976	3.904	3.056	2.665	2.432	2.274	2.160
175	1.654	1.974	3.895	3.048	2.656	2.423	2.266	2.151
200	1.653	1.972	3.888	3.041	2.650	2.417	2.259	2.144
9999999	1.645	1.960	3.841	2.996	2.605	2.372	2.214	2.099

Q.1. Consider the following 3 models.

Model A: $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$ Model B: $Y_i = \beta_1 X_i + \varepsilon_i$ Model C: $Y_i = \beta_0 + \varepsilon_i$

p.1.a. Derive the least squares estimators of the parameters for models A, B, and C (**Show all work**). Give the formulas for all 3 models below.

$$Q_A = \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 X_i)^2 \Rightarrow \frac{\partial Q_A}{\partial \beta_0} = 2\sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_i)(-1)$$

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$$\frac{\partial Q}{\partial \beta_1} = 2\sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_i)(-X_i)$$

(1) setting $\frac{\partial Q_A}{\partial \beta_0} = 0 \Rightarrow \sum_i Y_i = n\hat{\beta}_0 + \hat{\beta}_1 \sum_i X_i$ 6

(2) " $\frac{\partial Q}{\partial \beta_1} = 0 \Rightarrow \sum_i X_i Y_i = \hat{\beta}_0 \sum_i X_i + \hat{\beta}_1 \sum_i X_i^2$

$n(2) - (\sum_i X_i)(1) \Rightarrow n\sum_i X_i Y_i - \sum_i X_i \sum_i Y_i = 0 + \hat{\beta}_1 \left(n\sum_i X_i^2 - (\sum X_i)^2\right)$

$\sum_i (X_i - \bar{X})^2 = \sum_i (X_i^2 - 2\bar{X}X_i + \bar{X}^2) = \sum_i X_i^2 - 2\bar{X}\underbrace{\sum_i X_i}_{n\bar{X}} + n\bar{X}^2 = \sum_i X_i^2 - n\bar{X}^2 = \sum_i X_i^2 - \frac{(\sum X_i)^2}{n}$

similarly: $\sum_i (X_i - \bar{X})(Y_i - \bar{Y}) = \sum X_i Y_i - \frac{(\sum X_i)(\sum Y_i)}{n}$ 6

$$n\sum_i (X_i - \bar{X})(Y_i - \bar{Y}) = \hat{\beta}_1 \, n \sum_i (X_i - \bar{X})^2 \Rightarrow \hat{\beta}_1 = \frac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2}$$

From (1) : $n\hat{\beta}_0 = \sum_i Y_i - \hat{\beta}_1 \sum_i X_i \Rightarrow \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$ 4

$$Q_B = \sum_i (Y_i - \beta_1 X_i)^2 \Rightarrow \frac{\partial Q_B}{\partial \beta_1} = 2\sum_i (Y_i - \beta_1 X_i)(-X_i) = -2\sum_i (Y_i X_i - \beta_1 X_i^2)$$

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set $= 0 \Rightarrow \sum_i Y_i X_i = \hat{\beta}_1 \sum_i X_i^2 \Rightarrow \hat{\beta}_1 = \frac{\sum_i X_i Y_i}{\sum_i X_i^2}$

$$Q_C = \sum_i (Y_i - \beta_0)^2 \Rightarrow \frac{\partial Q_C}{\partial \beta_0} = 2\sum_i (Y_i - \beta_0)(-1) \overset{set}{=} 0 \Rightarrow \sum_i Y_i = n\hat{\beta}_0$$

$\Rightarrow \hat{\beta}_0 = \bar{Y}$ 6

Model A: $\hat{\beta}_1 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2}$ $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$ Model B: $\hat{\beta}_1 = \frac{\sum X_i Y_i}{\sum X_i^2}$ Model C: $\hat{\beta}_0 = \bar{Y}$

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p.1.b. A set of observations of average January temperature (Y, in °F) and latitude (X, in ° North latitude) for n = 4 Asian cities are given below. Compute the estimates and give the fitted values, residuals, and error sum of squares for models A and C in the following table (Hint: the means of Y and X are both integers).

$$\bar{Y} = \frac{1}{4}[79+57+28+8] = \frac{1}{4}[172] = 43$$

$$\bar{X} = \frac{1}{4}[14+29+38+43] = \frac{1}{4}[124] = 31$$

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City	$X-\bar{X}$	$Y-\bar{Y}$	$(X-\bar{X})^2$	$(Y-\bar{Y})(X-\bar{X})$
M	-17	36	289	-612
ND	-2	14	4	-28
S	7	-15	49	-105
V	12	-35	144	-420
			486	-1165

$$\hat{\beta}_1 = \frac{-1165}{486} = -2.397$$

$$\hat{\beta}_0 = 43 - (-2.397)(31) = 117.307$$

	Mod A		Mod B	
City	$\hat{y}$	e	$\hat{y}$	e
M	83.75	-4.75	43	36
ND	47.80	9.20	43	14
S	26.22	1.78	43	-15
V	14.24	-6.24	43	-35
		~0		0

$$SSE_A = (-4.75)^2 + 9.20^2 + 1.78^2 + (-6.24)^2 = 149.31$$

$$SSE_C = 36^2 + 14^2 + (-15)^2 + (-35)^2 = 2942$$

City	JanTemp (Y)	Latitude (X)	Yhat-Mod1 (8)	Yhat-Mod3 (4)	e - Mod1 (4)	e - Mod3 (4)
Manila	79	14	83.75	43	-4.75	36
New Delhi	57	29	47.80	43	9.20	14
Seoul	28	38	26.22	43	1.78	-15
Vladivostok	8	43	14.24	43	-6.24	-35
				SSE(Model)	149.31 (2)	2942 (2)

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Q.2. A study in Egypt related heights (Y, mm) to first metatarsal maximum length (X, mm) in a sample of n = 110 adult females. The sample means, standard deviations and correlations are given below. Complete the following table for the simple linear regression relating Height (Y) to right hand length (X).

$$\overline{X} = 69.06 \quad \overline{Y} = 1591.91 \quad s_X = \sqrt{\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n-1}} = 8.084 \quad s_Y = 87.025 \quad r_{XY} = 0.882$$

$SSR = .778(S_{YY}) =$

$r^2 = .882^2 = .778$ $\quad S_{xx} = 109(8.084)^2 = 7123.265$ $\quad S_{YY} = 109(87.025)^2 = 825495.218$

Regression Statistics						
R Square	.778	2				
Residual Std Error	41.193	3				
Observations	110	1				
ANOVA		3 each	2 each	2	2	
	df	SS	MS	F	F(.05)	
Regression 1	1	642235.280	642235.280	378.486	$\simeq$ 3.927	
Residual 1	108	183259.938	1696.851			
Total 1	109	825495.218	—			
		3 each	1/1	2/2	2/2	
	Coefficients	andard Err	t Stat	Lower	Upper	
Intercept 3	936.185	33.934	27.59	868.928	1003.442	67.257
X 3	9.495	0.488	19.46	8.528	10.462	.967

$$S_{xy} = (n-1) r_{xy} (s_x s_y) = 109(.882)(8.084)(87.025) = 67634.054$$

$$\hat{\beta}_1 = \frac{67634.054}{7123.265} = 9.495 \qquad \widehat{SE}\{\hat{\beta}_1\} = \sqrt{\frac{1696.851}{7123.265}} = .488$$

$$\hat{\beta}_0 = 1591.91 - 9.495(69.06) = 936.185 \qquad \widehat{SE}\{\hat{\beta}_0\} = \sqrt{1696.851\left(\frac{1}{110} + \frac{69.06^2}{7123.265}\right)}$$

$= 33.934$

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$t_{.025, 108} \simeq 1.982$

Q.3. Based on Model 2, defining $Q = \varepsilon'\varepsilon$: $\underset{\sim}{\varepsilon} = \underset{\sim}{Y} - X\underset{\sim}{\beta}$

p.3.a. Derive the least squares estimator of β. **Show all work** 12

$$Q = (\underset{\sim}{Y} - X\underset{\sim}{\beta})'(\underset{\sim}{Y} - X\underset{\sim}{\beta}) = \underset{\sim}{Y}'\underset{\sim}{Y} - \underset{\sim}{Y}'X\underset{\sim}{\beta} - \underset{\sim}{\beta}'X'\underset{\sim}{Y} + \underset{\sim}{\beta}'X'X\underset{\sim}{\beta}$$

$(\underset{\sim}{Y}'X\underset{\sim}{\beta})' = \underset{\sim}{\beta}'X'\underset{\sim}{Y}$, and dim = 1×1 $\Rightarrow -(\underset{\sim}{Y}'X\underset{\sim}{\beta} + \underset{\sim}{\beta}'X'\underset{\sim}{Y}) = -2\underset{\sim}{Y}'X\underset{\sim}{\beta}$

$$\Rightarrow Q = \underset{\sim}{Y}'\underset{\sim}{Y} - 2\underset{\sim}{Y}'X\underset{\sim}{\beta} + \underset{\sim}{\beta}'X'X\underset{\sim}{\beta}$$

$$\frac{\partial Q}{\partial \underset{\sim}{\beta}} = -2\underbrace{(\underset{\sim}{Y}'X)'}_{X'\underset{\sim}{Y}} + 2(X'X)\underset{\sim}{\beta} \overset{\text{set}}{=} 0 \Rightarrow X'X\underset{\sim}{\beta} = X'\underset{\sim}{Y}$$

$$\Rightarrow \hat{\underset{\sim}{\beta}} = (X'X)^{-1}X'\underset{\sim}{Y}$$

p.3.b. Give the following vectors in matrix form as matrix products involving **Y**: $\hat{\mathbf{Y}} = \begin{bmatrix} \hat{Y}_1 \\ \vdots \\ \hat{Y}_n \end{bmatrix}$, $\mathbf{e} = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$, $\overline{\mathbf{Y}} = \begin{bmatrix} \overline{Y} \\ \vdots \\ \overline{Y} \end{bmatrix}$

(4) $\hat{\underset{\sim}{Y}} = X\hat{\underset{\sim}{\beta}} = X(X'X)^{-1}X'\underset{\sim}{Y} = PY \quad (P = X(X'X)^{-1}X')$

(4) $\underset{\sim}{e} = \underset{\sim}{Y} - \hat{\underset{\sim}{Y}} = (I-P)\underset{\sim}{Y}$

(4) $\overline{\underset{\sim}{Y}} = \frac{1}{n}J_n\underset{\sim}{Y} \quad (\overline{Y} = \frac{1}{n}\underset{\sim}{1}'\underset{\sim}{Y} \Rightarrow \overline{\underset{\sim}{Y}} = \underset{\sim}{1}\frac{1}{n}\underset{\sim}{1}'\underset{\sim}{Y} = \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}'\underset{\sim}{Y} = \frac{1}{n}J\underset{\sim}{Y})$

p.3.c. Derive $E\{\hat{\mathbf{Y}}\}$, $V\{\hat{\mathbf{Y}}\}$, $V\{\mathbf{e}\}$ **Show all work**

NOTE: $PP = X\underbrace{(X'X)^{-1}X'X}_{I}(X'X)^{-1}X' = X(X'X)^{-1}X' = P$

$P' = (X(X'X)^{-1}X')' = X(\underbrace{(X'X)^{-1}}_{\text{symmetric}})'X' = P \Rightarrow P(I-P) = P - P = 0$

$(I-P)(I-P) = I - P - P + PP = I - P$

(8)
$$\Rightarrow E\{\hat{\underset{\sim}{Y}}\} = E\{P\underset{\sim}{Y}\} = PE\{\underset{\sim}{Y}\} = PX\beta = X\underbrace{(X'X)^{-1}X'X}_{I}\underset{\sim}{\beta} = X\underset{\sim}{\beta}$$

(8) $V\{\hat{\underset{\sim}{Y}}\} = V\{P\underset{\sim}{Y}\} = PV\{\underset{\sim}{Y}\}P' = P\sigma^2 IP' = \sigma^2 PP' = \sigma^2 PP = \sigma^2 P$

(8) $V\{\underset{\sim}{e}\} = V\{(I-P)\underset{\sim}{Y}\} = (I-P)V\{\underset{\sim}{Y}\}(I-P)' = (I-P)\sigma^2 I(I-P)$
$= \sigma^2(I-P)(I-P) = \sigma^2(I-P)$

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Q.4. For the simple regression model (scalar form, Model 1):

p.4.a. Give a_i such that $\hat{Y}_k = \sum_{i=1}^{n} a_i Y_i$

$$\hat{Y}_k = \hat{\beta}_0 + \hat{\beta}_1 X_k = \bar{Y} + \hat{\beta}_1 (X_k - \bar{X})$$

$$\Rightarrow \hat{Y}_k = \sum_{i=1}^{n} \left[\frac{1}{n} + \frac{(X_i - \bar{X})(X_k - \bar{X})}{S_{xx}} \right] Y_i$$

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p.4.b. Derive $E\left\{\hat{Y}_i\right\}$, $V\left\{\hat{Y}_i\right\}$, $\mathrm{COV}\left\{\hat{Y}_i, Y_i\right\}$ **Show all work**

$$E\{Y_i\} = \beta_0 + \beta_1 X_i \qquad V\{Y_i\} = \sigma^2 \qquad \mathrm{Cov}\{Y_i, Y_{i'}\} = 0 \quad i \neq i'$$

$$V\{\textstyle\sum_i a_i Y_i\} = \sum_i a_i^2 V\{Y_i\}$$

$$E\{\textstyle\sum_i a_i Y_i\} = \sum_i a_i E\{Y_i\}$$

$$\Rightarrow E\{\hat{Y}_k\} = \sum_{i=1}^{n} \left[\frac{1}{n} + \frac{(X_i - \bar{X})(X_k - \bar{X})}{S_{xx}} \right] (\beta_0 + \beta_1 X_i)$$

$$= \sum_{i=1}^{n} \left[\frac{1}{n}\beta_0 + \frac{1}{n}\beta_1 X_i + \frac{(X_i - \bar{X})(X_k - \bar{X})}{S_{xx}}\beta_0 + \frac{(X_i - \bar{X})(X_k - \bar{X}) X_i \beta_1}{S_{xx}} \right]$$

$$= \beta_0 + \beta_1 \bar{X} + \frac{\beta_0 (X_k - \bar{X})}{S_{xx}} \underbrace{\cancel{\textstyle\sum_i (X_i - \bar{X})}}_{0} + \frac{\beta_1 (X_k - \bar{X})}{\cancel{S_{xx}}} \frac{\cancel{\sum_i (X_i - \bar{X}) X_i}}{S_{xx}}$$

$$= \beta_0 + \beta_1 \bar{X} + \beta_1 (X_k - \bar{X}) = \underline{\beta_0 + \beta_1 X_k}$$

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12 $$\Rightarrow V\{\hat{Y}_k\} = \sum_{i=1}^{n} \left(\frac{1}{n} + \frac{(X_i - \bar{X})(X_k - \bar{X})}{S_{xx}} \right)^2 \sigma^2 = \sigma^2 \sum_{i=1}^{n} \left[\left(\frac{1}{n}\right)^2 + \frac{(X_i - \bar{X})^2 (X_k - \bar{X})^2}{S_{xx}^2} + 2\left(\frac{1}{n}\right) \frac{(X_i - \bar{X})(X_k - \bar{X})}{S_{xx}} \right]$$

$$= \sigma^2 \left[n\left(\frac{1}{n}\right)^2 + \frac{(X_k - \bar{X})^2}{S_{xx}^2} S_{xx} + \frac{2(X_k - \bar{X})}{n S_{xx}} \cancel{(X_i - \bar{X})} \right]$$ Sum to 0

$$= \sigma^2 \left[\frac{1}{n} + \frac{(X_k - \bar{X})^2}{S_{xx}} \right]$$

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$$\mathrm{Cov}\{Y_i, \hat{Y}_k\} = \sigma^2 [1] \left(\frac{1}{n} + \frac{(X_i - \bar{X})^2}{S_{xx}} \right] = V\{\hat{Y}_i\}$$

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Q.5. A simple linear regression model is fit involving fatigue in engineering processes. The response variable was log of number of cycles to failure (Y, in grams) and the predictor was shear stress (X). The matrix results are given below for Model 2.

X'X	
12.0000	21.8591
21.8591	39.9628

INV(X'X)	
23.0646	-12.6161
-12.6161	6.9258

X'Y
72.7248
131.7501

Y'Y
647.7123

p.5.a. Compute the least squares estimate $\hat{\boldsymbol{\beta}}$ and the unbiased estimate of σ^2 Hint: $\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{Y} = \mathbf{Y}'\mathbf{P}\mathbf{Y}$

$$\hat{\underset{\sim}{\beta}} = \begin{bmatrix} 23.0646(72.7248) - 12.6161(131.7501) \\ -12.6161(72.7248) + 6.9258(131.7501) \end{bmatrix} = \begin{bmatrix} 15.1960 \\ -5.0285 \end{bmatrix} \quad 8$$

$$Y'PY = 15.1960(72.7248) - 5.0285(131.7501) = 442.6207 \quad 4$$

$$SSE = Y'(I-P)Y = 647.7123 - 442.6207 = 205.0916$$

$$MSE = \frac{SSE}{n-2} = \frac{205.0916}{12-2} = 20.5092 \quad 4$$

p.5.b. Compute a 95% Confidence Interval for the mean log # of cycles to failure when shear stress is 1.80.

$$\hat{Y}_{1.80} = 15.1960 - 5.0285(1.80) = 6.1447 \quad 3$$

$$\hat{V}\{\hat{\beta}_0\} + 1.8^2 V\{\hat{\beta}_1\} + 2(1.8)\,Cov\{\hat{\beta}_0, \hat{\beta}_1\} = 20.5092\left[23.0646 + 1.8^2(6.9258) + 2(1.8)(-12.6161)\right]$$

$$= 20.5092[.0682] = 1.7685 \Rightarrow \hat{SE} = \sqrt{1.7685} = 1.3299 \quad 8$$

$$\Rightarrow 6.1447 \pm \underbrace{1.3299 \times 2.228}_{2.9630} \; 2 \; = \; \cancel{(\ldots,\ \ldots)}\ (3.1817,\ 9.1077) \quad 2$$

p.5.c. Compute a 95% Prediction Interval for the log # of cycles to failure when shear stress is 1.80 in a new experimental run.

$$\hat{Y} = 6.1447 \qquad SE = \sqrt{20.5092(1.0682)} = \sqrt{21.9079} = 4.6806 \quad 4$$

$$4.6806 \times 2.228 = 10.4284$$

$$6.1447 \pm 10.4284 = (-4.2837,\ 16.5731 \quad 2$$

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