1-Way Random Effects Models

Q.1. A 1-Way random effects model was fit, comparing copper concentrations among bottles of a Scotch whiskey brand for a given age. A random sample of 5 bottles was selected, and within each bottle 3 samples were taken, and copper concentration was measured. The following table gives the sample means and variances for the 5 bottles for the model.

\[ y_{ij} = \mu + a_i + e_{ij} \quad a_i \sim NID\left(0, \sigma_a^2\right) \quad e_{ij} \sim NID\left(0, \sigma_e^2\right) \quad \{a_i\} \perp \{e_{ij}\} \]

<table>
<thead>
<tr>
<th>Bottle</th>
<th>#reps</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>265</td>
<td>500</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>335</td>
<td>1300</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>270</td>
<td>200</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>355</td>
<td>1225</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>375</td>
<td>625</td>
</tr>
</tbody>
</table>

p.1.a. Compute the Among Bottle sum of squares, give its degrees of freedom, and Expected Mean Square

\[ SSA = \ldots \quad \text{df}_A = \ldots \quad \text{E}\{\text{MSA}\} = \ldots \]

p.1.b. Compute the Within Bottle sum of squares, give its degrees of freedom, and Expected Mean Square

\[ SSW = \ldots \quad \text{df}_W = \ldots \quad \text{E}\{\text{MSW}\} = \ldots \]

p.1.c. Test whether the variance of the bottle effects is 0.  H\(_0\): \(\sigma_a^2 = 0\)  H\(_A\): \(\sigma_a^2 > 0\)

Test Statistic: \ldots  Rejection Region: \ldots

p.1.d. Obtain Point estimates for: \(\hat{\sigma}_e^2\), \(\hat{\sigma}_a^2\), \(\hat{\rho}_I = \frac{\hat{\sigma}_a^2}{\hat{\sigma}_a^2 + \hat{\sigma}_e^2}\)

\[ \hat{\sigma}_e^2 = \ldots \quad \hat{\sigma}_a^2 = \ldots \quad \hat{\rho}_I = \ldots \]

Q.2. Consider the 1-Way Random Effects Model:

\[ y_{ij} = \mu + a_i + e_{ij} \quad i = 1, \ldots, g \quad j = 1, \ldots, n \quad a_i \sim NID\left(0, \sigma_a^2\right) \quad e_{ij} \sim NID\left(0, \sigma_e^2\right) \quad \{a_i\} \perp \{e\} \]

p.2.a. Based on rules involving linear functions of RVs, derive the following values:

\[ E(y_{ij}), \quad V(y_{ij}), \quad V(y_{i*}), \quad V(y_{*}), \quad \text{COV}(y_{ij}, y_{i*}) \]

p.2.b. In a population of individuals, 95% have mean values between 80 and 120. \(\rho_I\), the Intraclass Correlation is 0.80. Within individuals, 95% of individual observations lie within how many units from the individual mean?
Q.3. For the 1-Way Random Effects model, derive the Expected Mean Squares for Treatments and Error by completing the following steps. 

\[ Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \] 
\[ \alpha_i \sim \text{NID}(0, \sigma^2_{\alpha}) \] 
\[ \varepsilon_{ij} \sim \text{NID}(0, \sigma^2) \] 
\[ \{\alpha\} \perp \{\varepsilon\} \] 
\[ i = 1, ..., g; j = 1, ..., n \]

p.3.a. Derive: \( E\{Y_{ij}\}, \ V\{Y_{ij}\}, \ E\{Y_{i.}\}, \ V\{Y_{i.}\}, \ E\{Y_{..}\}, \ V\{Y_{..}\} \) SHOW ALL WORK.

p.3.b. Making use of p.3.a., derive: \( E\{\sum_{i=1}^{g} \sum_{j=1}^{n} Y_{ij}^2\}, \ E\{n \sum_{i=1}^{g} \bar{Y}_{i.}^2\}, \ E\{ng \bar{Y}_{..}^2\} \)

p.3.c. Making use of p.3.b., derive: \( E\{MS_{TRT}\} \) and \( E\{MS_{ERR}\} \)

Q.4. A 1-Way Random Effects model is fit, comparing readability scores among a random sample of \( g=6 \) Business/Economics columnists. A sample of \( n=3 \) essays were selected from each columnist and their readability was assessed based on the Flesch-Kincaid scale.

\[ Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \] 
\[ \alpha_i \sim N(0, \sigma^2_{\alpha}) \] 
\[ \varepsilon_{ij} \sim \text{NID}(0, \sigma^2) \] 
\[ \{\alpha\} \perp \{\varepsilon\} \] 
\[ i = 1, ..., g; j = 1, ..., n \]

p.4.a. Test \( H_0 : \sigma^2_{\alpha} = 0 \) \( H_A : \sigma^2_{\alpha} > 0 \)

Test Statistic: ______________________________   Rejection Region: ________________________________

p.4.b. Obtain a point estimate and an approximate 95% Confidence Interval for \( \sigma^2_{\alpha} \) (based on Satterthwaite’s Approx.)

Point Estimate: _________________    Approximate 95% Confidence Interval: _____________________________

Q.5. In a population of stock market analysts, the mean rate of return is 2%. Approximately 95% of all analysts have a personal mean return within 5% of the overall mean. Among individual stocks selected by a particular analyst, approximately 95% have returns within 7% of his/her mean.

\[ \sigma^2 = \frac{1}{g} \sum_{i=1}^{g} \left( \bar{Y}_{..} - \bar{Y}_{i.} \right)^2 \] 

Compute: \( \sigma^2 = \frac{1}{g} \sum_{i=1}^{g} \left( \bar{Y}_{..} - \bar{Y}_{i.} \right)^2 \)

Q.6. Jack runs a small firm with 3 employees. He plans to conduct an experiment to compare them with respect to the quantity of their output using a new drill press, observing each employee’s output over 5 sessions. Jill has a large firm with many employees. She plans on sampling 3 at random, and observing each employee’s output over 5 sessions.

p.6.a. Clearly state Jack’s (treatment effects) model, describing all elements and assumptions.

p.6.b. Derive (Showing all work) \( E\{\bar{Y}_{i.}\}, \ V\{\bar{Y}_{i.}\}, \ E\{\bar{Y}_{..}\}, \ V\{\bar{Y}_{..}\} \) for Jack’s model.
Clearly state Jill’s model, describing all elements and assumptions.

Derive (Showing all work) \( E\{\bar{Y}_{ij}\}, \ V\{\bar{Y}_{ij}\}, \ E\{\bar{Y}_{..}\}, \ V\{\bar{Y}_{..}\} \), for Jill’s model.

Q7. A wine company produces bottles of wine that have mean alcohol contents (percentages) that are normally distributed, with 95% of all bottle means between 10.5 and 13.5. For specimens within bottles, alcohol contents are normally distributed with 95% of specimens falling within 0.6 of the bottle mean. Based on the 1-way random effects model, give:

\[ \sigma^2_a = \]
\[ \sigma^2 = \]
\[ \rho = \]

Q8. An experiment is conducted to study variation in breaking strength of denim jeans in a large automated manufacturing facility. There are 1000s of machines. Each machine has a true mean breaking strength of jeans that it produces, and these means are normally distributed. Approximately 95% of the machines have means between 40 and 60. Within a given machine, approximately 95% of individual pairs of jeans lie within 6 of that machine’s mean. The model fit is:

\[ y_{ij} = \mu_i + \epsilon_{ij} \quad \mu_i \sim NID(\mu, \sigma_a^2) \quad \epsilon_{ij} \sim NID(0, \sigma^2) \quad \{\mu_i\} \perp \{\epsilon_{ij}\} \]

For this scenario, obtain the following parameters and measures:

\[ \mu = \ldots \quad \sigma_a^2 = \ldots \quad \sigma^2 = \ldots \quad \rho_i = \text{Corr}(Y_{ij}, Y_{i'j}) = \ldots \quad j \neq j' \]

Q9. Among a population of lakes, the mean adult fish lengths are normally distributed, with approximately 95% of the lake means lying between 50 and 70 centimeters. Within lakes, approximately 95% of the fish have lengths within 12 cm of the lake mean. Consider a 1-Way random effects model, where a sample of \( g \) lakes is selected and \( n \) fish are sampled from each lake.

\[ y_{ij} = \mu + \alpha_i + \epsilon_{ij} \quad i = 1,...,g; j = 1,...,n \quad \alpha_i \sim NID(0, \sigma_a^2) \quad \epsilon_{ij} \sim NID(0, \sigma^2) \quad \{\alpha_i\} \perp \{\epsilon_{ij}\} \]

p.9.a. Obtain \( \mu, \ \sigma_a^2, \ \sigma^2 \)

p.9.b. Give the mean and variance of \( \bar{Y}_{..} \)
Q.10. Consider the 1-Way Random Effects Model:

\[ y_{ij} = \mu + a_i + e_{ij} \quad i = 1, \ldots, g \quad j = 1, \ldots, n \quad a_i \sim NID(0, \sigma_a^2) \quad e_{ij} \sim NID(0, \sigma_e^2) \quad \{a\} \perp \{e\} \]

p.10.a. Based on rules involving linear functions of RVs, derive the following values:

\[ E(y_{ij}), \quad V(y_{ij}), \quad V(y_{i*}), \quad V(y_{**}), \quad COV(y_{ij}, y_{*j}) \]

p.10.b. In a population of individuals, 95% have mean values between 40 and 60. Within individuals, 95% of individual observations lie within 4 units from the individual mean. Compute \( \rho_i \), the Intraclass Correlation