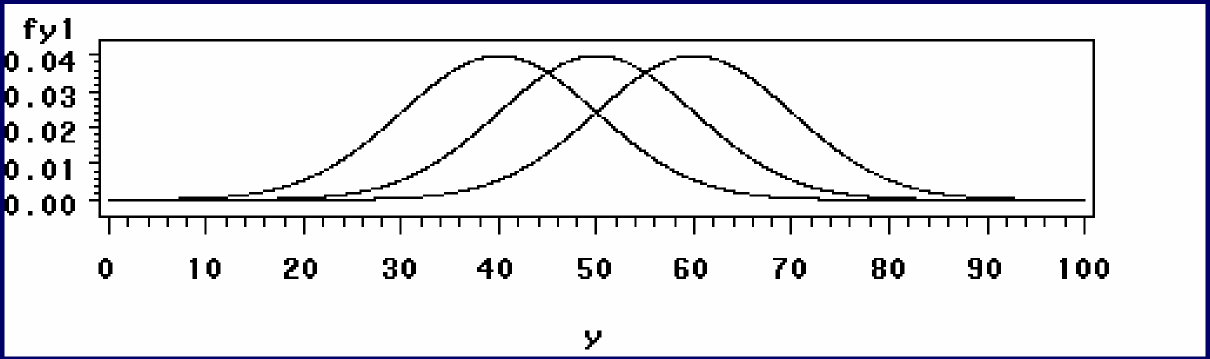
# 1-Way Analysis of Variance

- Setting:
  - Comparing $g > 2$ groups
  - Numeric (quantitative) response
  - Independent samples
- Notation (computed for each group):
  - Sample sizes: $n_1, \ldots, n_g$ ($N = n_1 + \ldots + n_g$)
  - Sample means: $\overline{Y}_1, \ldots, \overline{Y}_g$ $\left( \overline{Y} = \dfrac{n_1 \overline{Y}_1 + \cdots + n_g \overline{Y}_g}{N} \right)$
  - Sample standard deviations: $s_1, \ldots, s_g$

# 1-Way Analysis of Variance

- Assumptions for Significance tests:
  - The *g* distributions for the response variable are normal
  - The population standard deviations are equal for the *g* groups ($\sigma$)
  - Independent random samples selected from the *g* populations



# Within and Between Group Variation

- Within Group Variation: Variability among individuals within the same group. (*WSS*)
- Between Group Variation: Variability among group means, weighted by sample size. (*BSS*)

$$WSS = (n_1 - 1)s_1^2 + \cdots + (n_g - 1)s_g^2 \qquad df_W = N - g$$

$$BSS = n_1\left(\overline{Y}_1 - \overline{Y}\right)^2 + \cdots + n_g\left(\overline{Y}_g - \overline{Y}\right)^2 \qquad df_B = g - 1$$

- If the population means are all equal, $E(WSS/df_W) = E(BSS/df_B) = \sigma^2$

# Example: Policy/Participation in European Parliament

- Group Classifications: Legislative Procedures *(g*=4): (Consultation, Cooperation, Assent, Co-Decision)
- Units: Votes in European Parliament
- Response: Number of Votes Cast

| Legislative Procedure (i) | # of Cases ($n_i$) | Mean $\left(\overline{Y}_i\right)$ | Std. Dev ($s_i$) |
|---|---|---|---|
| Consultation | 205 | 296.5 | 124.7 |
| Cooperation | 88 | 357.3 | 93.0 |
| Assent | 8 | 449.6 | 171.8 |
| Codecision | 133 | 368.6 | 61.1 |

$$N = 205 + 88 + 8 + 133 = 434 \qquad \overline{Y} = \frac{205(296.5) + 88(357.3) + 8(449.6) + 133(368.6)}{434} = \frac{144845.5}{434} = 333.75$$

Source: R.M. Scully (1997). "Policy Influence and Participation in the European Parliament", *Legislative Studies Quarterly*, pp.233-252.

# Example: Policy/Participation in European Parliament

| i | n_i | Ybar_i | s_i | YBar_i-Ybar | BSS | WSS |
|---|---|---|---|---|---|---|
| 1 | 205 | 296.5 | 124.7 | -37.25 | 284450.313 | 3172218 |
| 2 | 88 | 357.3 | 93.0 | 23.55 | 48805.02 | 752463 |
| 3 | 8 | 449.6 | 171.8 | 115.85 | 107369.78 | 206606.7 |
| 4 | 133 | 368.6 | 61.1 | 34.85 | 161531.493 | 492783.7 |
| | | | | | **602156.605** | **4624072** |

$$BSS = 205(296.5 - 333.75)^2 + \cdots + 133(368.6 - 333.75)^2 = 602156.6 \qquad df_B = 4 - 1 = 3$$

$$WSS = (205 - 1)(124.7)^2 + \cdots + (133 - 1)(61.1)^2 = 4624072 \qquad df_W = 434 - 4 = 430$$

# $F$-Test for Equality of Means

- $H_0$: $\mu_1 = \mu_2 = \cdots = \mu_g$
- $H_A$: The means are not all equal

$$T.S.\ F_{obs} = \frac{BSS/(g-1)}{WSS/(N-g)} = \frac{BMS}{WMS}$$

$$R.R.: F_{obs} \geq F_{\alpha,g-1,N-g}$$

$$P = P(F \geq F_{obs})$$

- BMS and WMS are the Between and Within **Mean Squares**

# Example: Policy/Participation in European Parliament

- $H_0$: $\mu_1 = \mu_2 = \mu_3 = \mu_4$
- $H_A$: The means are not all equal

$$T.S.\ F_{obs} = \frac{BSS/(g-1)}{WSS/(N-g)} = \frac{602156.6/3}{4624072/430} = 18.67$$

$$R.R.: F_{obs} \geq F_{\alpha,g-1,N-g} = F_{.05,3,430} \approx 2.60$$

$$P = P(F \geq F_{obs} = 18.67) < P(F \geq 5.42) = .001$$

# Analysis of Variance Table

- Partitions the total variation into Between and Within Treatments (Groups)
- Consists of Columns representing: Source, Sum of Squares, Degrees of Freedom, Mean Square, *F*-statistic, *P*-value (computed by statistical software packages)

| Source of Variation | Sum of Squares | Degrres of Freedom | Mean Square | *F* |
|---|---|---|---|---|
| Between | *BSS* | *g*-1 | *BMS*=*BSS*/(*g*-1) | *F*=*BMS*/*WMS* |
| Within | *WSS* | *N*-*g* | *WMS*=*WSS*/(*N*-*g*) | |
| Total | *TSS* | *N*-1 | | |

# Estimating/Comparing Means

- Estimate of the (common) standard deviation:

$$\hat{\sigma} = \sqrt{\frac{WSS}{N-g}} = \sqrt{WMS} \qquad df = N - g$$

- Confidence Interval for $\mu_i$:

$$\overline{Y}_i \pm t_{\alpha/2, N-g} \frac{\hat{\sigma}}{\sqrt{n_i}}$$

- Confidence Interval for $\mu_i - \mu_j$ :

$$\left(\overline{Y}_i - \overline{Y}_j\right) \pm t_{\alpha/2, N-g}\, \hat{\sigma} \sqrt{\frac{1}{n_i} + \frac{1}{n_j}}$$

# Multiple Comparisons of Groups

- Goal: Obtain confidence intervals for all pairs of group mean differences.
- With $g$ groups, there are $g(g\text{-}1)/2$ pairs of groups.
- Problem: If we construct several (or more) 95% confidence intervals, the probability that they all contain the parameters ($\mu_i - \mu_j$) being estimated will be less than 95%
- Solution: Construct each individual confidence interval with a higher confidence coefficient, so that they will all be correct with 95% confidence

# Bonferroni Multiple Comparisons

- Step 1: Select an experimentwise error rate ($\alpha_E$), which is 1 minus the overall confidence level. For 95% confidence for all intervals, $\alpha_E$=0.05.
- Step 2: Determine the number of intervals to be constructed: $g(g$-1)/2
- Step 3: Obtain the comparisonwise error rate: $\alpha_C = \alpha_E/[g(g-1)/2]$
- Step 4: Construct (1- $\alpha_C$)100% CI's for $\mu_i$-$\mu_j$:

$$\left(\overline{Y}_i - \overline{Y}_j\right) \pm t_{\alpha_C/2, N-g}\, \hat{\sigma} \sqrt{\frac{1}{n_i} + \frac{1}{n_j}}$$

# Interpretations

- After constructing all $g(g-1)/2$ confidence intervals, make the following conclusions:
  - Conclude $\mu_i > \mu_j$ if CI is strictly positive
  - Conclude $\mu_i < \mu_j$ if CI is strictly negative
  - Do not conclude $\mu_i \neq \mu_j$ if CI contains 0
- Common graphical description.
  - Order the group labels from lowest mean to highest
  - Draw sequence of lines below labels, such that means that are not significantly different are "connected" by lines

# Example: Policy/Participation in European Parliament

- Estimate of the common standard deviation:

$$\hat{\sigma} = \sqrt{\frac{WSS}{N-g}} = \sqrt{\frac{4624072}{430}} = 103.7$$

- Number of pairs of procedures: 4(4-1)/2=6
- Comparisonwise error rate: $\alpha_C = .05/6 = .0083$
- $t_{.0083/2,430} \approx z_{.0042} \approx 2.64$

# Example: Policy/Participation in European Parliament

| Comparison | $\overline{Y}_i - \overline{Y}_j$ | $t\hat{\sigma}\sqrt{\frac{1}{n_i}+\frac{1}{n_j}}$ | Confidence Interval |
|---|---|---|---|
| Consult vs Cooperate | 296.5-357.3 = -60.8 | 2.64(103.7)(0.13)=35.6 | (-96.4 , -25.2)* |
| Consult vs Assent | 296.5-449.6 = -153.1 | 2.64(103.7)(0.36)=98.7 | (-251.8 , -54.4)* |
| Consult vs Codecision | 296.5-368.6 = -72.1 | 2.64(103.7)(0.11)=30.5 | (-102.6 , -41.6)* |
| Cooperate vs Assent | 357.3-449.6 = -92.3 | 2.64(103.7)(0.37)=101.1 | (-193.4 , 8.8) |
| Cooperate vs Codecision | 357.3-368.6 = -11.3 | 2.64(103.7)(0.14)=37.6 | (-48.9 , 26.3) |
| Assent vs Codecision | 449.6-368.6 = 81.0 | 2.64(103.7)(0.36)=99.7 | (-18.7 , 180.7) |

Consultation Cooperation Codecision Assent

Population mean is lower for consultation than all other procedures, no other procedures are significantly different.

# Regression Approach To ANOVA

- Dummy (Indicator) Variables: Variables that take on the value 1 if observation comes from a particular group, 0 if not.
- If there are *g* groups, we create *g*-1 dummy variables.
- Individuals in the "baseline" group receive 0 for all dummy variables.
- Statistical software packages typically assign the "last" ($g^{th}$) category as the baseline group
- Statistical Model: $E(Y) = \alpha + \beta_1 Z_1 + \ldots + \beta_{g-1} Z_{g-1}$
- $Z_i$ =1 if observation is from group i, 0 otherwise
- Mean for group i (i=1,...,g-1): $\mu_i = \alpha + \beta_i$
- Mean for group *g*: $\mu_g = \alpha$

# Test Comparisons

- $\mu_i = \alpha + \beta_i \quad \mu_g = \alpha \quad \Rightarrow \beta_i = \mu_i - \mu_g$

- 1-Way ANOVA: $H_0: \mu_1 = \ldots = \mu_g$
- Regression Approach: $H_0: \beta_1 = \ldots = \beta_{g-1} = 0$

- Regression *t*-tests: Test whether means for groups *i* and *g* are significantly different:
  – $H_0: \beta_i = \mu_i - \mu_g = 0$

# 2-Way ANOVA

- 2 nominal or ordinal factors are believed to be related to a quantitative response
- Additive Effects: The effects of the levels of each factor do not depend on the levels of the other factor.
- Interaction: The effects of levels of each factor depend on the levels of the other factor
- Notation: $\mu_{ij}$ is the mean response when factor A is at level *i* and Factor B at *j*

# Example - Thalidomide for AIDS

- Response: 28-day weight gain in AIDS patients
- Factor A: Drug: Thalidomide/Placebo
- Factor B: TB Status of Patient: $TB^+$/$TB^-$
- Subjects: 32 patients (16 $TB^+$ and 16 $TB^-$). Random assignment of 8 from each group to each drug). Data:
  - Thalidomide/$TB^+$: 9,6,4.5,2,2.5,3,1,1.5
  - Thalidomide/$TB^-$: 2.5,3.5,4,1,0.5,4,1.5,2
  - Placebo/$TB^+$: 0,1,-1,-2,-3,-3,0.5,-2.5
  - Placebo/$TB^-$: -0.5,0,2.5,0.5,-1.5,0,1,3.5

# ANOVA Approach

- Total Variation (*TSS*) is partitioned into 4 components:
  - Factor A: Variation in means among levels of A
  - Factor B: Variation in means among levels of B
  - Interaction: Variation in means among combinations of levels of A and B that are not due to A or B alone
  - Error: Variation among subjects within the same combinations of levels of A and B (Within SS)

# ANOVA Approach

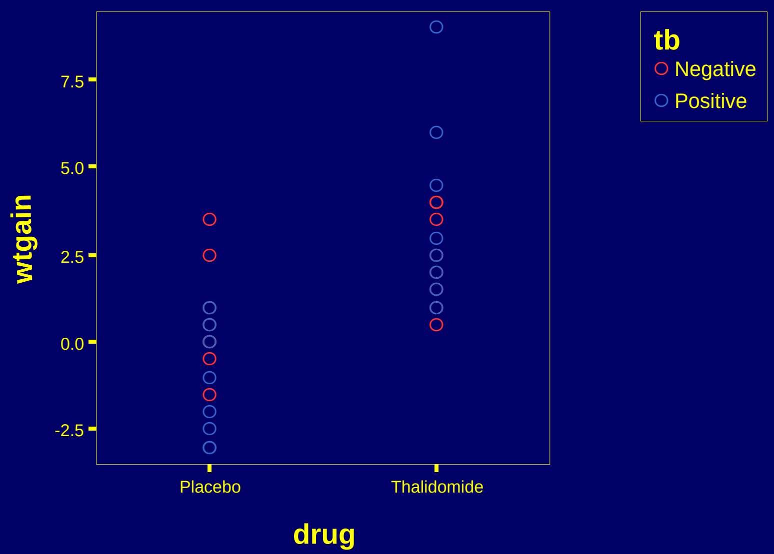
General Notation: Factor A has *a* levels, B has *b* levels

| Source | df | SS | MS | F |
|---|---|---|---|---|
| Factor A | a-1 | SSA | MSA=SSA/(a-1) | $F_A$=MSA/WMS |
| Factor B | b-1 | SSB | MSB=SSB/(b-1) | $F_B$=MSB/WMS |
| Interaction | (a-1)(b-1) | SSAB | MSAB=SSAB/[(a-1)(b-1)] | $F_{AB}$=MSAB/WMS |
| Error | N-ab | WSS | WMS=WSS/(N-ab) | |
| Total | N-1 | TSS | | |

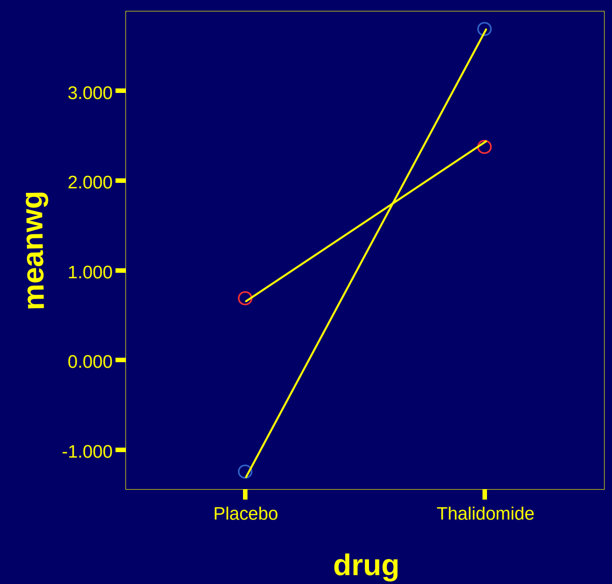
- Procedure:
  - Test $H_0$: No interaction based on the $F_{AB}$ statistic
  - If the interaction test is not significant, test for Factor A and B effects based on the $F_A$ and $F_B$ statistics

# Example - Thalidomide for AIDS

Individual Patients




Group Means




**Report**

WTGAIN

| GROUP | Mean | N | Std. Deviation |
|---|---|---|---|
| TB+/Thalidomide | 3.688 | 8 | 2.6984 |
| TB-/Thalidomide | 2.375 | 8 | 1.3562 |
| TB+/Placebo | -1.250 | 8 | 1.6036 |
| TB-/Placebo | .688 | 8 | 1.6243 |
| Total | 1.375 | 32 | 2.6027 |

# Example - Thalidomide for AIDS

**Tests of Between-Subjects Effects**

Dependent Variable: WTGAIN

| Source | Type III Sum of Squares | df | Mean Square | F | Sig. |
|---|---|---|---|---|---|
| Corrected Model | 109.688[a] | 3 | 36.563 | 10.206 | .000 |
| Intercept | 60.500 | 1 | 60.500 | 16.887 | .000 |
| DRUG | 87.781 | 1 | 87.781 | 24.502 | .000 |
| TB | .781 | 1 | .781 | .218 | .644 |
| DRUG * TB | 21.125 | 1 | 21.125 | 5.897 | .022 |
| Error | 100.313 | 28 | 3.583 | | |
| Total | 270.500 | 32 | | | |
| Corrected Total | 210.000 | 31 | | | |

a. R Squared = .522 (Adjusted R Squared = .471)

- There is a significant Drug*TB interaction ($F_{DT}$=5.897, P=.022)
- The Drug effect depends on TB status (and vice versa)

# Regression Approach

- General Procedure:
  - Generate $a$-1 dummy variables for factor A ($A_1,\ldots,A_{a-1}$)
  - Generate $b$-1 dummy variables for factor B ($B_1,\ldots,B_{b-1}$)

- Additive (No interaction) model:

$$E(Y) = \alpha + \beta_1 A_1 + \cdots + \beta_{a-1} A_{a-1} + \beta_a B_1 + \cdots + \beta_{a+b-2} B_{b-1}$$

Test for differences among levels of factor A : $H_0 : \beta_1 = \cdots = \beta_{a-1} = 0$

Test for differences among levels of factor B : $H_0 : \beta_a = \cdots = \beta_{a+b-2} = 0$

Tests based on fitting full and reduced models.

# Example - Thalidomide for AIDS

- Factor A: Drug with $a$=2 levels:
  - $D$=1 if Thalidomide, 0 if Placebo
- Factor B: TB with $b$=2 levels:
  - $T$=1 if Positive, 0 if Negative
- Additive Model:
- Population Means:
  - Thalidomide/TB$^+$: $\alpha+\beta_1+\beta_2$
  - Thalidomide/TB$^-$: $\alpha+\beta_1$
  - Placebo/TB$^+$: $\alpha+\beta_2$
  - Placebo/TB$^-$: $\alpha$

$$E(Y) = \alpha + \beta_1 D + \beta_2 T$$

- Thalidomide (vs Placebo Effect) Among TB$^+$/TB$^-$ Patients:
- TB$^+$: $(\alpha+\beta_1+\beta_2)-(\alpha+\beta_2) = \beta_1$ TB$^-$: $(\alpha+\beta_1)-\alpha = \beta_1$

# Example - Thalidomide for AIDS

- Testing for a Thalidomide effect on weight gain:
  - $H_0: \beta_1 = 0$ vs $H_A: \beta_1 \neq 0$ ($t$-test, since $a$-1=1)
- Testing for a $TB^+$ effect on weight gain:
  - $H_0: \beta_2 = 0$ vs $H_A: \beta_2 \neq 0$ ($t$-test, since $b$-1=1)
- SPSS Output: (Thalidomide has positive effect, TB None)

**Coefficients[a]**

| Model | | Unstandardized Coefficients | | Standardized Coefficients | t | Sig. |
|---|---|---|---|---|---|---|
| | | B | Std. Error | Beta | | |
| 1 | (Constant) | -.125 | .627 | | -.200 | .843 |
| | DRUG | 3.313 | .723 | .647 | 4.579 | .000 |
| | TB | -.313 | .723 | -.061 | -.432 | .669 |

a. Dependent Variable: WTGAIN

# Regression with Interaction

- Model with interaction (A has *a* levels, B has *b*):
  - Includes *a*-1 dummy variables for factor A main effects
  - Includes *b*-1 dummy variables for factor B main effects
  - Includes (*a*-1)(*b*-1) cross-products of factor A and B dummy variables
- Model:

$$E(Y) = \alpha + \beta_1 A_1 + \cdots + \beta_{a-1} A_{a-1} + \beta_a B_1 + \cdots + \beta_{a+b-2} B_{b-1} + \beta_{a+b-1}(A_1 B_1) + \cdots + \beta_{ab-1}(A_{a-1} B_{b-1})$$

As with the ANOVA approach, we can partition the variation to that attributable to Factor A, Factor B, and their interaction

# Example - Thalidomide for AIDS

- Model with interaction: $E(Y) = \alpha + \beta_1 D + \beta_2 T + \beta_3 (DT)$
- Means by Group:
  - Thalidomide/TB$^+$: $\alpha + \beta_1 + \beta_2 + \beta_3$
  - Thalidomide/TB$^-$: $\alpha + \beta_1$
  - Placebo/TB$^+$: $\alpha + \beta_2$
  - Placebo/TB$^-$: $\alpha$
- Thalidomide (vs Placebo Effect) Among TB$^+$ Patients:
  - $(\alpha + \beta_1 + \beta_2 + \beta_3) - (\alpha + \beta_2) = \beta_1 + \beta_3$
- Thalidomide (vs Placebo Effect) Among TB$^-$ Patients:
  - $(\alpha + \beta_1) - \alpha = \beta_1$
- Thalidomide effect is same in both TB groups if $\beta_3 = 0$

# Example - Thalidomide for AIDS

- SPSS Output from Multiple Regression:

**Coefficients[a]**

| Model | | Unstandardized Coefficients | | Standardized Coefficients | t | Sig. |
|---|---|---|---|---|---|---|
| | | B | Std. Error | Beta | | |
| 1 | (Constant) | .687 | .669 | | 1.027 | .313 |
| | DRUG | 1.688 | .946 | .329 | 1.783 | .085 |
| | TB | -1.937 | .946 | -.378 | -2.047 | .050 |
| | DRUGTB | 3.250 | 1.338 | .549 | 2.428 | .022 |

a. Dependent Variable: WTGAIN

We conclude there is a Drug*TB interaction (t=2.428, p=.022). Compare this with the results from the two factor ANOVA table

# 1- Way ANOVA with Dependent Samples (Repeated Measures)

- Some experiments have the same subjects (often referred to as blocks) receive each treatment.
- Generally subjects vary in terms of abilities, attitudes, or biological attributes.
- By having each subject receive each treatment, we can remove subject to subject variability
- This increases precision of treatment comparisons.

# 1- Way ANOVA with Dependent Samples (Repeated Measures)

- Notation: *g* Treatments, *b* Subjects, *N*=*gb*
- Mean for Treatment *i*: $\overline{T}_i$
- Mean for Subject (Block) *j*: $\overline{S}_j$
- Overall Mean: $\overline{Y}$

$$\text{Total Sum of Squares}: SSTO = \sum\left(Y - \overline{Y}\right)^2 \quad df_{TO} = N - 1$$

$$\text{Between Treatmentt SS}: SSTR = b\sum\left(\overline{T} - \overline{Y}\right)^2 \quad df_{TR} = g - 1$$

$$\text{Between Subject SS}: SSBL = g\sum\left(\overline{S} - \overline{Y}\right)^2 \quad df_{BL} = b - 1$$

$$\text{Error SS}: SSE = SSTO - SSTR - SSBL \quad df_E = (g-1)(b-1)$$

# ANOVA & F-Test

| Source | df | SS | MS | F |
|---|---|---|---|---|
| Treatments | g-1 | SSTR | MSTR=SSTR/(g-1) | F=MSTR/MSE |
| Blocks | b-1 | SSBL | MSBL=SSBL/(b-1) | |
| Error | (g-1)(b-1) | SSE | MSE=SSE/[(g-1)(b-1)] | |
| Total | gb-1 | SSTO | | |

$$H_0: \text{No Difference in Treatment Means}$$

$$H_A: \text{Differences in Trt Means Exist}$$

$$T.S.\ F_{obs} = \frac{MSTR}{MSE}$$

$$R.R.\ F_{obs} \geq F_{\alpha, g-1, (g-1)(b-1)}$$

$$P = P(F \geq F_{obs})$$

# Post hoc Comparisons (Bonferroni)

- Determine number of pairs of Treatment means ($g(g-1)/2$)
- Obtain $\alpha_C = \alpha_E/(g(g-1)/2)$ and $t_{\alpha_C/2,(g-1)(b-1)}$
- Obtain $\hat{\sigma} = \sqrt{MSE}$
- Obtain the "critical quantity": $t\hat{\sigma}\sqrt{\frac{2}{b}}$
- Obtain the simultaneous confidence intervals for all pairs of means (with standard interpretations):

$$\left(\overline{T}_i - \overline{T}_j\right) \pm t\hat{\sigma}\sqrt{\frac{2}{b}}$$

# Repeated Measures ANOVA

- Goal: compare *g* treatments over *t* time periods
- Randomly assign subjects to treatments (Between Subjects factor)
- Observe each subject at each time period (Within Subjects factor)
- Observe whether treatment effects differ over time (interaction, Within Subjects)

# Repeated Measures ANOVA

- Suppose there are $N$ subjects, with $n_i$ in the $i^{th}$ treatment group.
- Sources of variation:
  - Treatments ($g$-1 df)
  - Subjects within treatments aka Error1 ($N$-$g$ df)
  - Time Periods ($t$-1 df)
  - Time x Trt Interaction (($g$-1)($t$-1) df)
  - Error2 (($N$-$g$)($t$-1) df)

# Repeated Measures ANOVA

| Source | df | SS | MS | F |
|---|---|---|---|---|
| **Between Subjects** | | | | |
| Treatment | g-1 | SSTrt | MSTrt=SSTrt/(g-1) | MSTrt/MSE1 |
| Subj(Trt) = Error1 | N-g | SSE1 | MSE1=SSE1/(N-g) | |
| **Within Subjects** | | | | |
| Time | *t*-1 | SSTi | MSTi=SSTi/(*t*-1) | MSTi/MSE2 |
| TimexTrt | (*t*-1)(*g*-1) | SSTiTrt | MSTiTrt=SSTiTrt/((*t*-1)(*g*-1)) | MSTiTrt/MSE2 |
| Time*Subj(Trt)=Error2 | (*N*-*g*)(*t*-1) | SSE2 | MSE2=SSE2/((*N*-*g*)(*t*-1)) | |

To Compare pairs of treatment means (assuming no time by treatment interaction, otherwise they must be done within time periods and replace *tn* with just *n*):

$$\left(\overline{T}_i - \overline{T}_j\right) \pm t_{\alpha/2,N-g}\sqrt{MSE1\left(\frac{1}{tn_i}+\frac{1}{tn_j}\right)}$$