Chapter 6 Solutions

6.1 $\bar{d} = \begin{bmatrix} -9.36 \\ 13.27 \end{bmatrix}$ $S_d = \begin{bmatrix} 199.26 & 88.38 \\ 88.38 & 418.61 \end{bmatrix}$ $n=11$, $p=2$

$$\begin{vmatrix} 199.26-\lambda & 88.38 \\ 88.38 & 418.61-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (199.26-\lambda)(418.61-\lambda) - 88.38^2 = 0$$

$$\Rightarrow 83412.2286 - 617.87\lambda + \lambda^2 - 7811.0244 = 0$$

$$\Rightarrow \lambda^2 - 617.87\lambda + 75601.2042 = 0$$

$$\lambda = \frac{-(-617.87) \pm \sqrt{(-617.87)^2 - 4(1)(75601.2042)}}{2(1)}$$

$$= \frac{617.87 \pm 281.71}{2} = (449.79, 168.08)$$

$$\begin{bmatrix} 199.26 & 88.38 \\ 88.38 & 418.61 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} = 449.79 \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 199.26\, e_{11} + 88.38\, e_{12} &= 449.79\, e_{11} \\ 88.38\, e_{11} + 418.61\, e_{12} &= 449.79\, e_{12} \end{aligned}$$

$$\Rightarrow 250.53\, e_{11} = 88.38\, e_{12} \Rightarrow e_{11} = 2.835^{-1} e_{12}$$

$$\Rightarrow e_1^* = \begin{bmatrix} 1 \\ 2.835 \end{bmatrix} \Rightarrow e_1 = \begin{bmatrix} 1/3.006 \\ 2.835/3.006 \end{bmatrix} = \begin{bmatrix} .333 \\ .943 \end{bmatrix}$$

6.1 Continued

$$199.26 e_{21} + 88.38 e_{22} = 168.08 e_{21}$$
$$88.38 e_{21} + 418.61 e_{22} = 449.79 e_{22}$$

$$-31.18 e_{21} = 88.38 e_{22} \Rightarrow e_{21} = -2.835 e_{22}$$

$$\underset{\sim}{e}_2^{*} = \begin{bmatrix} -2.835 \\ 1 \end{bmatrix} \Rightarrow \underset{\sim}{e}_2 \simeq \begin{bmatrix} -.943 \\ .333 \end{bmatrix}$$

$$\frac{p(n-1)}{n(n-p)} F_{p,n-p}(\alpha) = \frac{2(10)}{11(9)} 4.256 = 0.860$$

half-lengths

$$1: \sqrt{\lambda_1} \sqrt{\frac{p(n-1)}{n(n-p)} F_{p,n-p}(\alpha)} = \sqrt{449.79(.860)} = 19.67$$

$$2: \sqrt{\lambda_2} \sqrt{\text{——}} = \sqrt{168.08(.860)} = 12.02$$

ellipsoid centered @ $\bar{d} = \begin{bmatrix} -9.36 \\ 13.27 \end{bmatrix}$

w/ major axis $\underset{\sim}{e}_1 = \begin{bmatrix} .333 \\ .943 \end{bmatrix}$ $\underset{\sim}{e}_2 = \begin{bmatrix} -.943 \\ .333 \end{bmatrix}$

$$19.67 \underset{\sim}{e}_1 = \begin{bmatrix} 6.55 \\ 18.55 \end{bmatrix} \qquad 12.02 \underset{\sim}{e}_2 = \begin{bmatrix} -11.33 \\ 4.00 \end{bmatrix}$$

Major axis

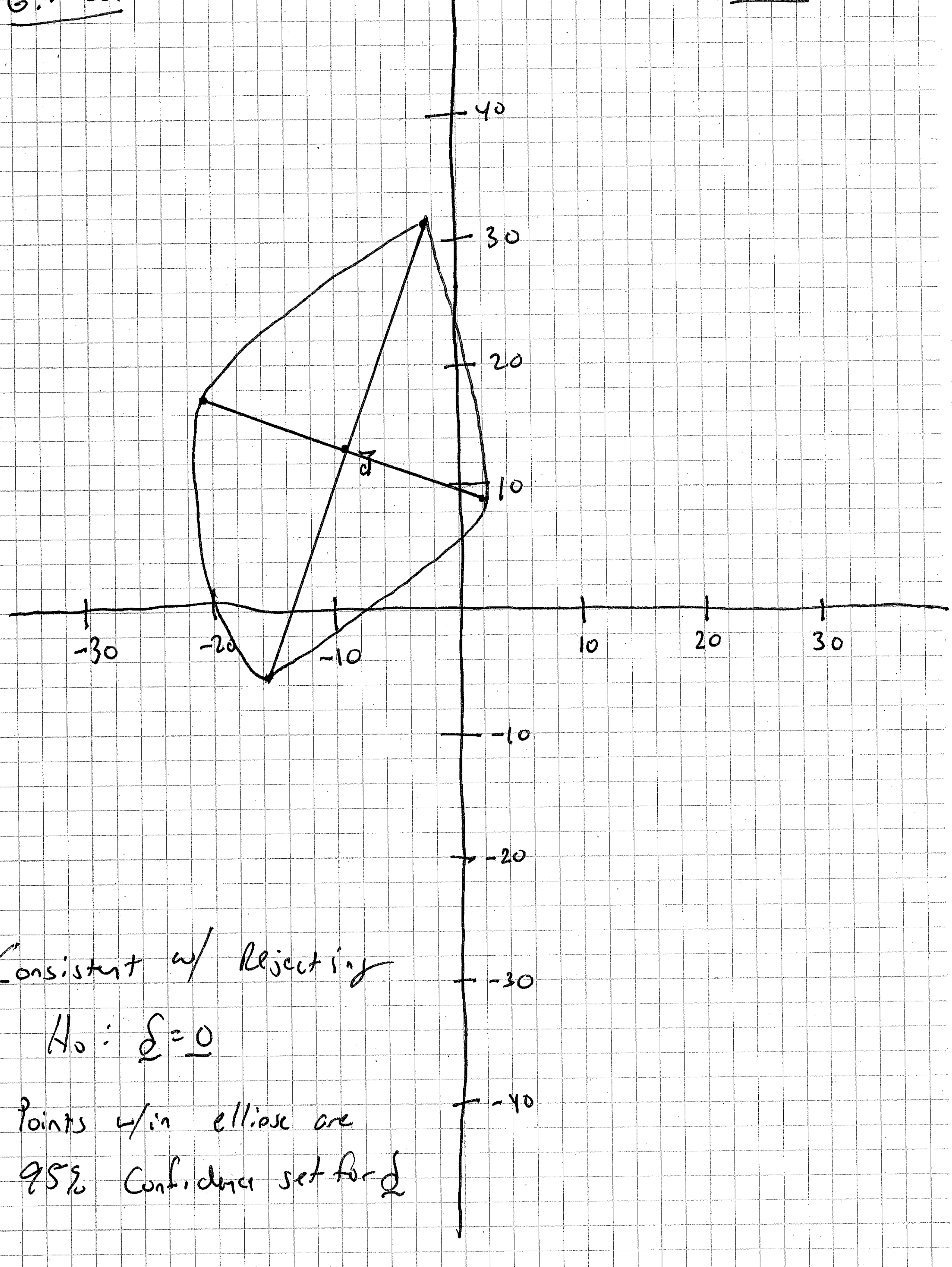
$$\begin{bmatrix} -9.36 + 6.55 = -2.81 \\ 13.27 + 18.55 = 31.82 \end{bmatrix} \begin{bmatrix} -9.36 - 6.55 = -15.91 \\ 13.27 - 18.66 = -5.39 \end{bmatrix}$$

Minor

$$\begin{bmatrix} -20.69 \\ 17.27 \end{bmatrix}, \begin{bmatrix} 1.97 \\ 9.27 \end{bmatrix}$$

6.1 Continued

6.3



Consistent w/ Rejecting

$H_0 : \underline{\delta} = \underline{0}$

Points w/in ellipse are
95% Confidence set for $\underline{\delta}$

6.2 $m=2$ $\quad t_{n-1}\left(\frac{\alpha}{2m}\right) = t_{10}\left(\frac{.05}{2(2)}\right) = 2.634$

$$\delta_1: \ \bar{d}_1 \pm 2.634\sqrt{\frac{s_{d_{11}}}{n}} \equiv -9.36 \pm 2.634\sqrt{\frac{199.26}{11}}$$
$$\equiv -9.36 \pm 11.21 \equiv (-20.47, 1.85)$$

$$\delta_2: \ 13.27 \pm 2.634\sqrt{\frac{418.61}{11}} \equiv 13.27 \pm 16.25 \equiv (-2.98, 29.52)$$

	Simultaneous lengths	Bonferroni lengths
δ_1	26.2	22.42
δ_2	37.96	32.50

6.5 $p=3$ $\quad n=40$ $\quad \bar{\underset{\sim}{X}} = \begin{bmatrix} 46.1 \\ 57.3 \\ 50.4 \end{bmatrix}$ $\quad S = \begin{bmatrix} 101.3 & 63.0 & 71.0 \\ 63.0 & 80.2 & 55.6 \\ 71.0 & 55.6 & 97.4 \end{bmatrix}$

$q=3$

a) $H_0: \mu_1 = \mu_2 = \mu_3 \Rightarrow C\mu = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$C\bar{\underset{\sim}{X}} = \begin{bmatrix} -46.1 + 57.3 \\ -46.1 + 50.4 \end{bmatrix} = \begin{bmatrix} 11.2 \\ 4.3 \end{bmatrix}$$

$$CSC' = \begin{bmatrix} -38.3 & -17.2 & -15.4 \\ -30.3 & -7.4 & 26.4 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 21.1 & 22.9 \\ 22.9 & 56.7 \end{bmatrix}$$

$$(CSC')^{-1} = \frac{1}{21.1(56.7) - 22.9^2}\begin{bmatrix} 56.7 & -22.9 \\ -22.9 & 21.1 \end{bmatrix} = \frac{1}{671.96}\begin{bmatrix} 56.7 & -22.9 \\ -22.9 & 21.1 \end{bmatrix}$$

6.5 Continued

$$T^2 = \frac{40}{671.96}\begin{bmatrix}11.2 & 4.3\end{bmatrix}\begin{bmatrix}56.7 & -22.9 \\ -22.9 & 21.1\end{bmatrix}\begin{bmatrix}11.2 \\ 4.3\end{bmatrix}$$

$$= \frac{40}{671.96}\begin{bmatrix}536.57 & -165.75\end{bmatrix}\begin{bmatrix}11.2 \\ 4.3\end{bmatrix}$$

$$= \frac{40}{671.96}(5296.859) = 315.31$$

$$RR: T^2 \geq \frac{(40-1)(3-1)}{40-3+1} F_{3-1,\,40-3+1}(.05) = \underbrace{2.053(3.245)}_{6.662}$$

Reject Ho

b) $c_1'\bar{x} = 11.2 \quad c_1'Sc_1 = 21.1 \quad \sqrt{\frac{(n-1)(q-1)}{n-q+1} F_{q-1,n-q+1}(\alpha)} = \sqrt{6.662} = 2.581$

$c_2'\bar{x} = 4.3 \quad c_2'Sc_2 = 56.7$

$c_1'\mu: \; 11.2 \pm 2.581\sqrt{\frac{21.1}{40}} \equiv 11.2 \pm 1.875 \equiv (9.325, 13.075)$

$c_2'\mu: \; 4.3 \pm 2.581\sqrt{\frac{56.7}{40}} \equiv 4.3 \pm 3.073 \equiv (1.227, 7.373)$

6.6

6.7 $\bar{X}_1 = \begin{bmatrix} 204.4 \\ 556.6 \end{bmatrix}$ $S_1 = \begin{bmatrix} 13825.3 & 23823.4 \\ 23823.4 & 73107.4 \end{bmatrix}$ $n_1 = 45$

$\bar{X}_2 = \begin{bmatrix} 130.0 \\ 355.0 \end{bmatrix}$ $S_2 = \begin{bmatrix} 8632.0 & 19616.7 \\ 19616.7 & 55964.5 \end{bmatrix}$ $n_2 = 55$

$$S_{pooled} = \begin{bmatrix} 10963.7 & 21505.5 \\ 21505.5 & 63661.3 \end{bmatrix}$$

$$\bar{X}_1 - \bar{X}_2 = \begin{bmatrix} 74.4 \\ 201.6 \end{bmatrix} \qquad \frac{1}{n_1} + \frac{1}{n_2} = \frac{1}{45} + \frac{1}{55} = .0404$$

$$\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pooled} = \begin{bmatrix} 443.0 & 868.9 \\ 868.9 & 2572.2 \end{bmatrix}$$

$$\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pooled}\right]^{-1} = \frac{1}{443.0(2572.2) - 868.9^2} \begin{bmatrix} 2572.2 & -868.9 \\ -868.9 & 443.0 \end{bmatrix}$$

$$= \frac{1}{384485.8} \begin{bmatrix} 2572.2 & -868.9 \\ -868.9 & 443.0 \end{bmatrix}$$

$$T.S.\quad T^2 = \frac{1}{384485.8} \begin{bmatrix} 74.4 & 201.6 \end{bmatrix} \begin{bmatrix} 2572.2 & -868.9 \\ -868.9 & 443.0 \end{bmatrix} \begin{bmatrix} 74.4 \\ 201.6 \end{bmatrix}$$

$$= \frac{1}{384485.8} \begin{bmatrix} 16201.4 & 24662.6 \end{bmatrix} \begin{bmatrix} 74.4 \\ 201.6 \end{bmatrix} = \frac{6177372.4}{384485.8} = 16.07$$

$$RR: T^2 \geq \frac{(45+55-2)(2)}{(45+55-2-1)} F_{2,45+55-2-1}^{(.05)} = 2.021(3.090) = 6.245$$

Reject $H_0: \mu_1 = \mu_2$

6.7 continued

$$\hat{a} \propto S_{pooled}^{-1}(\bar{X}_1 - \bar{X}_2)$$

$$\propto \begin{bmatrix} 2572.2 & -868.9 \\ -868.9 & 443.0 \end{bmatrix} \begin{bmatrix} 74.4 \\ 201.6 \end{bmatrix} = \begin{bmatrix} 16201.4 \\ 24662.6 \end{bmatrix} \Big\}$$

Total ~~Mean~~ ~~20432~~ 40864

$$\Rightarrow \hat{a} \propto \begin{bmatrix} .40 \\ .60 \end{bmatrix}$$

$$\Rightarrow .40[\bar{X}_{11} - \bar{X}_{21}] + .60[\bar{X}_{12} - \bar{X}_{22}]$$

6.8 Trt 1: $\bar{X}_1 = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ Trt 2: $\bar{X}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ Trt 3: $\bar{X}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\bar{X} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

a)

$$X_{1j} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \left(\begin{bmatrix} 6 \\ 8 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right) + \left(X_{1j} - \begin{bmatrix} 6 \\ 8 \end{bmatrix} \right)$$

$$X_{11}: \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$X_{12}: \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$X_{13}: \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$X_{14}: \begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$X_{15}: \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_{2j} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right) + \left(X_{2j} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right)$$

6.8 continued

$$X_{21}: \begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}-2\\-1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$$

$$X_{22}: \begin{bmatrix}1\\6\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}-2\\-1\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix}$$

$$X_{23}: \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}-2\\-1\end{bmatrix} + \begin{bmatrix}0\\-1\end{bmatrix}$$

$$X_{3j}: \begin{bmatrix}4\\5\end{bmatrix} + \left(\begin{bmatrix}3\\2\end{bmatrix} - \begin{bmatrix}4\\5\end{bmatrix}\right) + \left(X_{3j} - \begin{bmatrix}3\\2\end{bmatrix}\right)$$

$$X_{31}: \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}-1\\-3\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}$$

$$X_{32}: \begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}-1\\-3\end{bmatrix} + \begin{bmatrix}2\\-1\end{bmatrix}$$

$$X_{33}: \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}-1\\-3\end{bmatrix} + \begin{bmatrix}0\\-1\end{bmatrix}$$

$$X_{34}: \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}-1\\-3\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}$$

b) $$SSCP_{TRT} = \sum_{\ell=1}^{3} n_\ell [\bar{x}_\ell - \bar{x}][\bar{x}_\ell - \bar{x}]'$$

$$= 5\begin{bmatrix}2\\3\end{bmatrix}[2\ 3] + 3\begin{bmatrix}-2\\-1\end{bmatrix}[-2\ -1] + 4\begin{bmatrix}-1\\-3\end{bmatrix}[-1\ -3]$$

$$= 5\begin{bmatrix}4 & 6\\6 & 9\end{bmatrix} + 3\begin{bmatrix}4 & 2\\2 & 1\end{bmatrix} + 4\begin{bmatrix}1 & 3\\3 & 9\end{bmatrix} = \begin{bmatrix}36 & 48\\48 & 84\end{bmatrix}$$

$$SSCP_{ERR} = \sum_{\ell=1}^{3}\sum_{j=1}^{n_\ell} [y_{\ell j} - \bar{y}_\ell][y_{\ell j} - \bar{y}_\ell]'$$

Trt 1: $$\begin{bmatrix}0\\-1\end{bmatrix}[0\ -1] + \begin{bmatrix}-1\\1\end{bmatrix}[-1\ 1] + \begin{bmatrix}2\\-2\end{bmatrix}[2\ -2] + \begin{bmatrix}-2\\1\end{bmatrix}[-2\ 1] + \begin{bmatrix}1\\1\end{bmatrix}[1\ 1]$$

$$= \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}4 & -4\\-4 & 4\end{bmatrix} + \begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix} + \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}10 & -6\\-6 & 8\end{bmatrix}$$

6.8b Continued.

Trt 2: $\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix}\begin{bmatrix}-1 & 2\end{bmatrix} + \begin{bmatrix}0\\-1\end{bmatrix}\begin{bmatrix}0 & -1\end{bmatrix}$

$$= \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}1 & -2\\-2 & 4\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & -3\\-3 & 6\end{bmatrix}$$

Trt 3: $\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} + \begin{bmatrix}2\\-1\end{bmatrix}\begin{bmatrix}2 & -1\end{bmatrix} + \begin{bmatrix}0\\-1\end{bmatrix}\begin{bmatrix}0 & -1\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}$

$$= \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}6 & -4\\-4 & 4\end{bmatrix}$$

$$\Rightarrow SSCP_{Err} = \begin{bmatrix}10 & -6\\-6 & 8\end{bmatrix} + \begin{bmatrix}2 & -3\\-3 & 6\end{bmatrix} + \begin{bmatrix}6 & -4\\-4 & 4\end{bmatrix} = \begin{bmatrix}18 & -13\\-13 & 18\end{bmatrix}$$

MANOVA Table

Source	df	SSCP
Trt	$3-1=2$	$B = \begin{bmatrix}36 & 48\\48 & 84\end{bmatrix}$
Error	$12-3=9$	$W = \begin{bmatrix}18 & -13\\-13 & 18\end{bmatrix}$
TOTAL	$12-1=11$	$B+W = \begin{bmatrix}54 & 35\\35 & 102\end{bmatrix}$

c) ~~p=2 g=3~~

$$\Lambda^* = \frac{|W|}{|B+W|} \qquad |W| = 18^2 - (-13)^2 = 155 \qquad |B+W| = 54(102) - 35^2 = 4283$$

$$\Rightarrow \Lambda^* = \frac{155}{4283} = .0362 \qquad \sqrt{\Lambda^*} = .1903$$

6.8.c. continued

$P=2 \quad g=3 \quad N=12$

① $\left(\frac{N-g-1}{g-1}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2(g-1),2(N-g-1)}$

$\frac{12-3-1}{3-1}\left(\frac{1-.1903}{.1903}\right) = 17.019 \qquad F_{4,16}(.01) = 4.773$

② $\left(\frac{N-p-2}{p}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2p,2(N-p-2)}$ Same result

③ Bartlett's Method

$-\left(N-1-\frac{p+g}{2}\right)\ln\Lambda^* \geq \chi^2_{p(g-1)}(\alpha)$

$= -\left(12-1-\frac{2+3}{2}\right)\ln(.0362) = -8.5(-3.319) = 28.21$

$\chi^2_4(.01) = 13.277$

6.9 $X_j = \begin{bmatrix} x_{1j1} \\ \vdots \\ x_{1jp} \\ \hdashline x_{2j1} \\ \vdots \\ x_{2jp} \end{bmatrix} \qquad C = \left[I_p \,\vdots\, -I_p \right]$

$CX_j = \begin{bmatrix} 1x_{1j1} + 0x_{1j2} + \cdots + 0x_{1jp} - 1x_{2j1} + 0x_{2j2} + \cdots + 0x_{2jp} \\ \vdots \\ 0x_{1j1} + 0x_{1j2} + \cdots + 1x_{1jp} + 0x_{2j1} + 0x_{2j2} + \cdots + -1x_{2jp} \end{bmatrix} = \begin{bmatrix} x_{1j1} - x_{2j1} \\ \vdots \\ x_{1jp} - x_{2jp} \end{bmatrix}$

6.9 Continued

$\Rightarrow C \underset{\sim}{X}_j = \underset{\sim}{d}_j$

$$\bar{X} = \frac{1}{n} \sum_j X_j = \begin{bmatrix} \bar{X}_{11} \\ \vdots \\ \bar{X}_{1p} \\ \vdots \\ \bar{X}_{21} \\ \vdots \\ \bar{X}_{2p} \end{bmatrix} \qquad C\bar{X} = \begin{bmatrix} \bar{X}_{11} - \bar{X}_{21} \\ \vdots \\ \bar{X}_{1p} - \bar{X}_{2p} \end{bmatrix}$$

$$\bar{d} = \frac{1}{n} \sum_{j=1}^{n} d_j = \frac{1}{n} \begin{bmatrix} \sum_j (X_{1j1} - X_{2j1}) \\ \vdots \\ \sum_j (X_{1jp} - X_{2jp}) \end{bmatrix} = \begin{bmatrix} \bar{X}_{11} - \bar{X}_{21} \\ \vdots \\ \bar{X}_{1p} - \bar{X}_{2p} \end{bmatrix}$$

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \hat{V}\{\underset{\sim}{X}_1\} & \hat{Cov}\{\underset{\sim}{X}_1, \underset{\sim}{X}_2\} \\ \hat{Cov}\{\underset{\sim}{X}_2, \underset{\sim}{X}_1\} & \hat{V}\{\underset{\sim}{X}_2\} \end{bmatrix}$$

$$\cancel{CS = \hat{V}\{X_{11}\} - \hat{Cov}\{X_{11}, X_{21}\} \quad \hat{Cov}\{X_{11}, X_{12}\} - \hat{Cov}\{X}$$

$$CS = \begin{bmatrix} S_{11} - S_{21} & \vdots & S_{12} - S_{22} \end{bmatrix}$$

$$CSC' = \begin{bmatrix} S_{11} - S_{21} - S_{12} + S_{22} \end{bmatrix} = \begin{bmatrix} S_{11} + S_{22} - 2S_{12} \end{bmatrix}$$

$$S_{11} = \frac{1}{n-1} \sum_j (\underset{\sim}{X}_{1j} - \bar{X}_1)(X_{1j} - \bar{X}_1)' \qquad S_{22} = \frac{1}{n-1} \sum_j (X_{2j} - \bar{X}_2)(X_{2j} - \bar{X}_2)'$$

$$S_{12} = \frac{1}{n-1} \sum_j (X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2)' \qquad S_{21} = \frac{1}{n-1} \sum_j (X_{2j} - \bar{X}_2)(X_{1j} - \bar{X}_1)'$$

6.9 Continued

$$S_d = \frac{1}{n-1}\left[\sum_j (d_j - \bar{d})(d_j - \bar{d})'\right]$$

$$= \frac{1}{n-1}\sum_j \left[(X_{1j} - X_{2j}) - (\bar{X}_1 - \bar{X}_2)\right]\left[(X_{1j} - X_{2j}) - (\bar{X}_1 - \bar{X}_2)\right]'$$

$$= \frac{1}{n-1}\sum_j \left[(X_{1j} - \bar{X}_1) - (X_{2j} - \bar{X}_2)\right]\left[(X_{1j} - \bar{X}_1) - (X_{2j} - \bar{X}_2)\right]'$$

$$= \frac{1}{n-1}\sum_j \Big[(X_{1j} - \bar{X}_1)(X_{1j} - \bar{X}_1)' - (X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2)'$$
$$- (X_{2j} - \bar{X}_2)(X_{1j} - \bar{X}_1)' + (X_{2j} - \bar{X}_2)(X_{2j} - \bar{X}_2)'\Big]$$

$$= S_{11} - S_{12} - S_{21} + S_{22} = S_{11} + S_{22} - 2S_{12}$$

6.10 univariate

$$X_{\ell j} = \bar{x} + (\bar{x}_\ell - \bar{x}) + (x_{\ell j} - \bar{x}_\ell)$$

$$(\bar{x}_1 - \bar{x})u_1 = \begin{pmatrix} (\bar{x}_1 - \bar{x})\underset{\sim}{1}_{n_1} \\ \underset{\sim}{0}_{n_2} \\ \vdots \\ \underset{\sim}{0}_{n_g} \end{pmatrix} \qquad \sum_\ell (\bar{x}_\ell - \bar{x})u_\ell = \begin{bmatrix} (\bar{x}_1 - \bar{x})\underset{\sim}{1}_{n_1} \\ (\bar{x}_2 - \bar{x})\underset{\sim}{1}_{n_2} \\ \vdots \\ (\bar{x}_g - \bar{x})\underset{\sim}{1}_{n_g} \end{bmatrix}$$

$$(\bar{x}\underset{\sim}{1})'\sum_\ell (\bar{x}_\ell - \bar{x})u_j = \bar{x}\left[\sum_{\ell=1}^{g} n_\ell(\bar{x}_\ell - \bar{x})\right]$$

$$= \bar{x}\left[\sum_{\ell=1}^{g} n_\ell \bar{x}_\ell - N\bar{x}\right] = \bar{x}\left(\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell} x_{j\ell} - N\frac{\sum_\ell\sum_j x_{\ell j}}{N}\right] = 0$$

6.11 $L(\mu_1, \mu_2, \Sigma) = L(\mu_1, \Sigma) L(\mu_2, \Sigma)$

$$L(\mu_1, \Sigma) = 2\pi^{-n_1 p/2} |\Sigma|^{-n_1/2} \times$$
$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(\sum_{j=1}^{n_1} (X_{1j} - \bar{X}_1)(X_{1j} - \bar{X}_1)' + n_1(\bar{X}_1 - \mu_1)(\bar{X}_1 - \mu_1)'\right)\right]\right.$$

$$L(\mu_2, \Sigma) = 2\pi^{-n_2 p/2} |\Sigma|^{-n_2/2} \times$$
$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(\sum_{j=1}^{n_2} (X_{2j} - \bar{X}_2)(X_{2j} - \bar{X}_2)' + n_2(\bar{X}_2 - \mu_2)(\bar{X}_2 - \mu_2)'\right)\right]\right.$$

$$L(\mu_1, \mu_2, \sigma) = 2\pi^{-(n_1+n_2)p/2} |\Sigma|^{-(n_1+n_2)/2}$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(\sum_{\ell=1}^{2}\left[\sum_{j=1}^{n_\ell} (X_{\ell j} - \bar{X}_\ell)(X_{\ell j} - \bar{X}_\ell)' + n_\ell(\bar{X}_\ell - \mu_\ell)(\bar{X}_\ell - \mu_\ell)'\right]\right)\right]\right\}$$

Last term $= \exp\left\{-\frac{1}{2}\left[\sum_\ell n_\ell (\bar{X}_\ell - \mu_\ell)' \Sigma^{-1} (\bar{X}_\ell - \mu_\ell)\right]\right\}$

$\Rightarrow$ ML estimates for μ_1, μ_2 are $\bar{X}_1, \bar{X}_2$

<u>Result 4.10</u>

$$\frac{1}{|\Sigma|^b} e^{-tr(\Sigma^{-1}B)/2} \le \frac{1}{|B|^b} (2b)^{pb} e^{-pb}$$

Let $b = \frac{n_1 + n_2}{2}$ $\quad B = \sum_{\ell=1}^{2} \sum_{j=1}^{n_\ell} (X_{\ell j} - \bar{X}_\ell)(X_{\ell j} - \bar{X}_\ell)'$

6.11 Continued

$\Rightarrow$ Max occurs @ $\hat{\Sigma} = \frac{1}{2b} B$

$$\Rightarrow \hat{\Sigma} = \frac{1}{n_1+n_2} \sum_{l=1}^{2} \sum_{j=1}^{n_l} (x_{lj} - \bar{x}_l)(x_{lj} - \bar{x}_l)'$$

$$= \frac{1}{n_1+n_2} \left[(n_1-1) S_1 + (n_2-1) S_2 \right]$$

$$S_{pooled} = \frac{(n_1-1) S_1 + (n_2-1) S_2}{n_1+n_2-2}$$

$$\Rightarrow \hat{\Sigma} = \frac{n_1+n_2-2}{n_1+n_2} S_{pooled}$$

6.12

$\underset{\sim}{\mu}_1' = [\mu_{11} \cdots \mu_{1p}]$ and $\underset{\sim}{\mu}_2' = [\mu_{21} \cdots \mu_{2p}]$ are parallel $\Rightarrow$ $\mu_{1i} - \mu_{2i} = c$ $\quad i = 1, \ldots, p$.

a) Linear profiles $\Rightarrow$ (*) $\mu_{li} - \mu_{l,i-1} = \mu_{l,i-1} - \mu_{l,i-2}$ $\quad l = 1,2 \ \forall i$

$$\underset{\sim}{\mu}_1 + \underset{\sim}{\mu}_2 = \begin{bmatrix} \mu_{11} + \mu_{21} \\ \vdots \\ \mu_{1p} + \mu_{2p} \end{bmatrix}$$

$$(*) \Rightarrow \underset{\sim}{0} = \left\{ \begin{array}{l} \left[(\mu_{13} + \mu_{23}) - (\mu_{12} + \mu_{22}) \right] - \left[(\mu_{12} + \mu_{22}) - (\mu_{11} + \mu_{21}) \right] \\ \vdots \\ \left[(\mu_{1p} + \mu_{2p}) - (\mu_{1,p-1} + \mu_{2,p-1}) \right] - \left[(\mu_{1,p-1} + \mu_{2,p-1}) + (\mu_{1,p-2} + \mu_{2,p-2}) \right] \end{array} \right\}$$

6.12a Continued

$$\Rightarrow \begin{bmatrix} (\mu_{13}+\mu_{23}) - 2(\mu_{12}+\mu_{22}) + (\mu_{11}+\mu_{21}) \\ \vdots \\ (\mu_{1p}+\mu_{2p}) - 2(\mu_{1,p-1}+\mu_{2,p-1}) + (\mu_{1,p-2}+\mu_{2,p-2}) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ & & \vdots & & & & & \\ & & & & & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & & & \end{bmatrix}$$

b) Reject H_0 if $T^2 = \left[C(\bar{X}_1+\bar{X}_2)\right]'\left[\left(\frac{1}{n_1}+\frac{1}{n_2}\right)C S_p C'\right]^{-1}$
$\times\, C(\bar{X}_1+\bar{X}_2) > c^2$

w/ $c^2 = \frac{(n_1+n_2-2)(p-2)}{n_1+n_2-p+1} F_{p-2,\, n_1+n_2-p+1}$

$n_1 = n_2 = 30 \qquad p = 4$

$$\bar{X}_1 = \begin{bmatrix} 6.4 \\ 6.8 \\ 7.3 \\ 7.0 \end{bmatrix} \quad \bar{X}_2 = \begin{bmatrix} 4.3 \\ 4.9 \\ 5.3 \\ 5.1 \end{bmatrix} \quad S_p = \begin{bmatrix} .61 & .26 & .07 & .16 \\ .26 & .64 & .17 & .14 \\ .07 & .17 & .81 & .03 \\ .16 & .14 & .03 & .31 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \qquad \bar{X}_1 + \bar{X}_2 = \begin{bmatrix} 10.7 \\ 11.7 \\ 12.6 \\ 12.1 \end{bmatrix}$$

6.12 b Continued

$$C(\bar{X}_1 + \bar{X}_2) = \begin{bmatrix} 10.7 - 2(11.7) + 12.6 \\ 11.7 - 2(12.6) + 12.1 \end{bmatrix} = \begin{bmatrix} -0.1 \\ -1.4 \end{bmatrix}$$

$$C S_p C' = \begin{bmatrix} 0.16 & -0.85 & 0.54 & -0.09 \\ 0.28 & 0.44 & -1.42 & 0.39 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 & -2.02 \\ -2.02 & 3.67 \end{bmatrix}$$

$$\left(\frac{1}{n_1} + \frac{1}{n_2}\right) C S_p C' = \frac{2}{30} \begin{bmatrix} 2.40 & -2.02 \\ -2.02 & 3.67 \end{bmatrix}$$

$$[\ -\ ]^{-1} = 15 \frac{1}{2.40(3.67) - (-2.02)^2} \begin{bmatrix} 3.67 & 2.02 \\ 2.02 & 2.40 \end{bmatrix}$$

$$= 3.1729 \begin{bmatrix} 3.67 & 2.02 \\ 2.02 & 2.40 \end{bmatrix}$$

$$T^2 = 3.1729 \begin{bmatrix} -0.1 & 1.4 \end{bmatrix} \begin{bmatrix} 3.67 & 2.02 \\ 2.02 & 2.40 \end{bmatrix} \begin{bmatrix} -0.1 \\ 1.4 \end{bmatrix}$$

$$= 3.1729 \begin{bmatrix} 2.461 & 3.158 \end{bmatrix} \begin{bmatrix} -0.1 \\ 1.4 \end{bmatrix} = 3.1729(4.1751) = 13.247$$

RR: $\cancel{T^2 > \frac{(30+30-2)(4-2)}{}}$

6.12b continued

$$RR:\ T^2 \geq \frac{(30+30-2)(4-2)}{30+30-4+1} F_{4-2,\,30+30-4+1}(.05)$$

$$= 2.035(3.159) = 6.429$$ Reject Ho.

6.17