Chapter 4 Problems

4.2. $\underset{\sim}{\mu}_X = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\sigma_{12} = .5\sqrt{2}\sqrt{1} = 0.7071$ $\Sigma_X = \begin{bmatrix} 2 & .7071 \\ .7071 & 1 \end{bmatrix}$

a) $f(x_1, x_2) = \dfrac{1}{2\pi\sqrt{2(1)(1-.5^2)}} \times$

$$\exp\left\{ -\frac{1}{2(1-.5^2)}\left[\left(\frac{x_1}{\sqrt{2}}\right)^2 + \left(\frac{x_2-2}{\sqrt{1}}\right)^2 - 2(.5)\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{x_2-2}{\sqrt{1}}\right)\right]\right\}$$

b) $(\underset{\sim}{x} - \underset{\sim}{\mu})' \Sigma_X^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}_X) =$

$$\begin{bmatrix} x_1 & x_2 - 2 \end{bmatrix} \frac{1}{2-0.5}\begin{bmatrix} 1 & -.7071 \\ -.7071 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} .6667x_1 - .4714(x_2-2) & -.4714x_1 + 1.3333(x_2-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 - 2 \end{bmatrix}$$

$$= .6667x_1^2 - 2(.4714)x_1(x_2-2) + 1.3333(x_2-2)^2$$

$$= .6667x_1^2 + 1.3333x_2^2 + 1.8856x_1 + x_2(-5.3332)$$

$$+ x_1x_2(-.9428) + 5.3332 = Q \qquad \chi_2^2(.50) = 1.3863$$

$$\{\{x_1, x_2\} : Q \le 1.3863\}$$

4.2

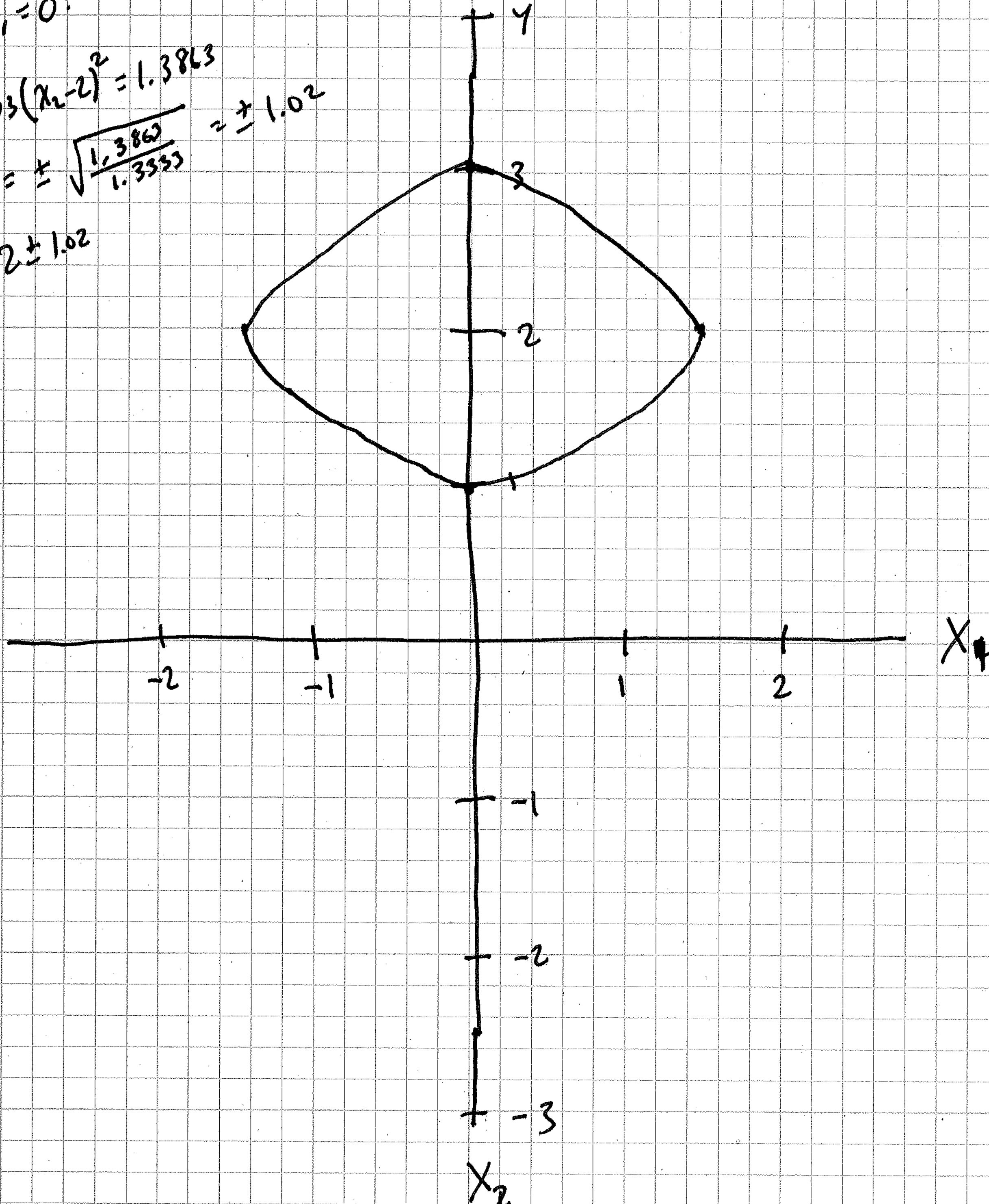
@ $X_2 = 2$: $Q = .6667\, x_1^2 = 1.3863 \Rightarrow x_1 = \pm\sqrt{\frac{1.3863}{.6667}}$
$= \pm 1.44$

@ $X_1 = 0$:

$Q = 1.3333(x_2 - 2)^2 = 1.3863$

$\Rightarrow x_2 - 2 = \pm\sqrt{\frac{1.3863}{1.3333}} = \pm 1.02$

$\Rightarrow x_2 = 2 \pm 1.02$



4.3 $X \sim N_3(\underset{\sim}{\mu}_x, \Sigma_x)$ $\quad \underset{\sim}{\mu}_x = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$ $\quad \Sigma_x = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

a) $Cov\{X_1, X_2\} = -2 \Rightarrow X_1, X_2$ not independent

b) $Cov\{X_2, X_3\} = 0 \Rightarrow X_1, X_3$ independent

c) $\left.\begin{array}{l} \text{"}\underset{\sim}{X}_1\text{"} = \begin{matrix} X_1 \\ X_2 \end{matrix} \\ \text{"}X_2\text{"} = X_3 \end{array}\right\}$ $\quad \Sigma_{XX} = \left[\begin{array}{cc:c} 1 & -2 & 0 \\ -2 & 5 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right]$ $\quad \Sigma_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$

(X_1, X_2) indep. of X_3

d) $\dfrac{X_1 + X_2}{2}, X_3$ $\quad a' = [\tfrac{1}{2} \ \tfrac{1}{2}]$ $\quad b = [1]$

$$Cov\{a'X_1, b'X_2\} = \underset{\sim}{a}' Cov\{X_1, X_2\} \underset{\sim}{b} = a' \Sigma_{12} b$$

$$= [\tfrac{1}{2} \ \ \tfrac{1}{2}] \begin{bmatrix} 0 \\ 0 \end{bmatrix} [1] = 0$$ independent

e) $Cov\{X_2, X_2 - \tfrac{5}{2} X_1 - X_3\}$

$$= Cov\{X_2, X_2\} - \tfrac{5}{2} Cov\{X_2, X_1\} - Cov\{X_2, X_3\}$$

$$= 5 - \tfrac{5}{2}(-2) - 0 = 5 + 5 = 10$$ Not independent

4.4 $\underset{\sim}{X} \sim N_3(\underset{\sim}{\mu}_x, \Sigma_x)$ $\underset{\sim}{\mu}_x = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ $\Sigma_x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

a)

$E\{3X_1 - 2X_2 + X_3\} = 3(2) - 2(-3) + 1(1) = 13$

$V\{3X_1 - 2X_2 + X_3\} = 9(1) + 4(3) + 1(2) + 2(3)(-2)(1)$

$+ 2(3)(1)(1) + 2(-2)(1)(2) = 9 + 12 + 2 - 12 + 6 - 8 = 9$

$\Rightarrow 3X_1 - 2X_2 + X_3 \sim N(13, 9)$

4.5. $\mu_x = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\Sigma_x = \begin{bmatrix} 2 & .7071 \\ .7071 & 1 \end{bmatrix}$

a)

$E\{X_1 | X_2 = x_2\} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_2)$

$= 0 + .7071 \left(\frac{1}{1}\right)(x_2 - 2) = .7071 x_2 - 1.4142$

$V\{X_1 | X_2\} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} = 2 - .7071\left(\frac{1}{1}\right)(.7071) = 1.500$

$\Rightarrow X_1 | X_2 = x_2 \sim N(.7071 x_2 - 1.4142, 1.50)$

b) $\begin{bmatrix} X_2 \\ X_1 \\ X_3 \end{bmatrix}$ $\mu_x = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ $\Sigma_x = \left[\begin{array}{c:cc} 3 & 1 & 2 \\ \hdashline 1 & 1 & 1 \\ 2 & 1 & 2 \end{array}\right]$ $[X_2 | X_1 = x_1, X_3 = x_3]$

$E\{X_2 | X_1 = x_1, X_3 = x_3\} = 1 + [1\ 2] \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x_1 - (-3) \\ x_3 - 4 \end{bmatrix}$

$1 + [1\quad 2] \frac{1}{2-1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 + 3 \\ x_3 - 4 \end{bmatrix}$

4.5b continued

$$= 1 + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1+3 \\ x_3-4 \end{bmatrix} = 1 + x_3 - 4 = X_3 - 3$$

$$V\{X_2 \mid X_1 = x_1, X_3 = x_3\} = 3 - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= 3 - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 - 2 = 1$$

$$X_2 \mid X_1 = x_1, X_3 = x_3 \sim N(x_3 - 3, 1)$$

c) $\cancel{\mu_X = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \quad \Sigma_X =}$

$$X = \begin{bmatrix} X_3 \\ X_1 \\ X_2 \end{bmatrix} \quad \mu_X = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \Sigma_X = \left[\begin{array}{c|cc} 2 & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 3 \end{array}\right]$$

$$E\{X_3 \mid X_1 = x_1, X_2 = x_2\} = 1 + \begin{bmatrix} 1 & 2 \end{bmatrix} \frac{1}{3-1} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 - (-3) \end{bmatrix}$$

$$= 1 + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 + 3 \end{bmatrix} = 1 + \frac{1}{2}(x_1 - 2) + \frac{1}{2}(x_2 + 3)$$

$$= \frac{3}{2} + \frac{1}{2} x_1 + \frac{1}{2} x_2$$

$$V\{X_3 \mid X_1 = x_1, X_2 = x_2\} = 2 - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= 2 - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 - \frac{3}{2} = \frac{1}{2}$$

$$\Rightarrow X_3 \mid X_1 = x_1, X_2 = x_2 \sim N\left(\frac{3}{2} + \frac{1}{2} x_1 + \frac{1}{2} x_2, \frac{1}{2}\right)$$

4.6

4.6

$\underset{\sim}{X} \sim N_3(\underset{\sim}{\mu}_x, \Sigma_x)$ $\underset{\sim}{\mu}_x = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ $\Sigma_x = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

a) $Cov\{X_1, X_2\} = 0 \Rightarrow$ independent

b) $Cov\{X_1, X_3\} = -1 \Rightarrow$ not independent

c) $Cov\{X_2, X_3\} = 0 \Rightarrow$ independent

d) (X_1, X_3) and X_2 $\Sigma_{x^*} = \left[\begin{array}{cc:c} 4 & -1 & 0 \\ -1 & 2 & 0 \\ \hdashline 0 & 0 & 5 \end{array}\right]$

$\underset{\sim}{X}^* = \begin{pmatrix} X_1 \\ X_3 \\ \hdashline X_2 \end{pmatrix}$ $\Sigma_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (X_1, X_3)$ and X_2 independent

e) $Cov\{X_1, X_1 + 3X_2 - 2X_3\}$

$= Cov\{X_1, X_1\} + 3\, Cov\{X_1, X_2\} - 2\, Cov\{X_1, X_3\}$

$= 4 + 3(0) - 2(-1) = 6$ not independent

4.16 $X_1, \dots, X_4$ iid $N_p(\underline{\mu}, \Sigma_x)$

$$V_1 = \frac{1}{4}X_1 - \frac{1}{4}X_2 + \frac{1}{4}X_3 - \frac{1}{4}X_4$$

$$E\{V_1\} = \frac{1}{4}\underline{\mu}_x - \frac{1}{4}\underline{\mu}_x + \frac{1}{4}\underline{\mu}_x - \frac{1}{4}\underline{\mu}_x = \underline{0}$$

$$V\{V_1\} = \frac{1}{16}(4)\Sigma_x = \frac{1}{4}\Sigma_x$$

$$V_2 = \frac{1}{4}X_1 + \frac{1}{4}X_2 - \frac{1}{4}X_3 - \frac{1}{4}X_4$$

$$E\{V_2\} = 0 \qquad V\{V_2\} = \frac{1}{4}\Sigma_x$$

$$Cov\{V_1, V_2\} = Cov\left\{\frac{1}{4}X_1 - \frac{1}{4}X_2 + \frac{1}{4}X_3 - \frac{1}{4}X_4, \frac{1}{4}X_1 + \frac{1}{4}X_2 - \frac{1}{4}X_3 - \frac{1}{4}X_4\right\}$$

$$= \frac{1}{16}V\{X_1\} - \frac{1}{16}V\{X_2\} - \frac{1}{16}V\{X_3\} + \frac{1}{16}V\{X_4\} = 0$$

Note: $Cov\{X_i, X_j\} = 0 \quad \forall\ i \neq j$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \sim N\left[\begin{pmatrix} \underline{0} \\ \underline{0} \end{pmatrix}, \begin{pmatrix} \frac{1}{4}\Sigma_x & 0 \\ 0 & \frac{1}{4}\Sigma_x \end{pmatrix}\right]$$

4.17 $\underset{\sim}{X}_i \sim \text{iid } N_p(\underset{\sim}{\mu}_x, \Sigma_x)$

$$E\left\{\tfrac{1}{5}X_1 + \tfrac{1}{5}X_2 + \tfrac{1}{5}X_3 + \tfrac{1}{5}X_4 + \tfrac{1}{5}X_5\right\} = 5\left(\tfrac{1}{5}\right)\underset{\sim}{\mu}_x = \underset{\sim}{\mu}_x$$

$$V\{\cdot\} = 5\left(\tfrac{1}{25}\right)\Sigma_x = \tfrac{1}{5}\Sigma_x$$

$$E\{X_1 - X_2 + X_3 - X_4 + X_5\} = \underset{\sim}{\mu}_x - \underset{\sim}{\mu}_x + \underset{\sim}{\mu}_x - \underset{\sim}{\mu}_x + \underset{\sim}{\mu}_x = \underset{\sim}{\mu}_x$$

$$V\{\cdot\} = 5(\pm 1)^2\Sigma_x = 5\Sigma_x$$

$$\mathrm{Cov}\{\cdot\,,\,\cdot\} = \tfrac{1}{5}(1)\Sigma_x + \tfrac{1}{5}(-1)\Sigma_x + \tfrac{1}{5}(1)\Sigma_x + \tfrac{1}{5}(-1)\Sigma_x + \tfrac{1}{5}(1)\Sigma_x$$

$$= \tfrac{1}{5}\Sigma_x$$

4.19 $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_{20}$ iid $N_6(\mu_x, \Sigma_x)$

a) $(\underset{\sim}{X}_1 - \underset{\sim}{\mu}_x)'\Sigma_x^{-1}(\underset{\sim}{X}_1 - \underset{\sim}{\mu}_x) \sim \chi^2_6$

b) $\bar{X} \sim N_6\left(\underset{\sim}{\mu}_x, \tfrac{1}{20}\Sigma_x\right)$

$\sqrt{n}(\bar{X} - \underset{\sim}{\mu}) \sim N_6(\underset{\sim}{0}, \Sigma_x)$

c) $(n-1)S \sim$ Wishart w/ $20-1=19$ df

4.21 $X_1, \dots, X_{60} \sim \text{iid } N_4(\underset{\sim}{\mu}_x, \Sigma_x)$

a) $\bar{X} \sim N_4\left(\underset{\sim}{\mu}_x, \frac{1}{60}\Sigma_x\right)$

b) $(\underset{\sim}{X}_1 - \underset{\sim}{\mu}_x)'\Sigma_x^{-1}(\underset{\sim}{X}_1 - \underset{\sim}{\mu}_x) \sim \chi^2_4$

c) $n(\bar{\underset{\sim}{X}} - \underset{\sim}{\mu}_x)'\Sigma_x^{-1}(\bar{\underset{\sim}{X}} - \underset{\sim}{\mu}_x) \sim \chi^2_4$

d) $n(\bar{\underset{\sim}{X}} - \underset{\sim}{\mu}_x)'S^{-1}(\underset{\sim}{X} - \underset{\sim}{\mu}_x) \dot{\sim} \chi^2_4$

4.22 $\underset{\sim}{X}_1, \dots, \underset{\sim}{X}_{75} \sim \text{iid } (\underset{\sim}{\mu}_x, \Sigma_x)$

a) $\bar{X} \dot{\sim} N\left(\underset{\sim}{\mu}_x, \frac{1}{75}\Sigma_x\right)$

b) $n(\bar{\underset{\sim}{X}} - \underset{\sim}{\mu}_x)'S^{-1}(\bar{\underset{\sim}{X}} - \underset{\sim}{\mu}_x) \dot{\sim} \chi^2_p$

Problem 4.29

4.10

R Program

```
tX <- t(X)
statdist <- rep(0,n)

for (i in 1:n) {
statdist[i] <- t(tX[,i]-xbar) %*% solve(S) %*% (tX[,i]-xbar)
}

sum(statdist <= qchisq(.50,2))/n
```

R Output

```
> (xbar <- as.matrix(colMeans(X),ncol=1))
         [,1]
NO2 10.047619
O3   9.404762
>
> n <- nrow(X)
>
> (S <- cov(X))
          NO2        O3
NO2 11.363531  3.126597
O3   3.126597 30.978513
>
> tX <- t(X)
> statdist <- rep(0,n)
>
> for (i in 1:n) {
+ statdist[i] <- t(tX[,i]-xbar) %*% solve(S) %*% (tX[,i]-xbar)
+ }
>
> sum(statdist <= qchisq(.50,2))/n
[1] 0.6190476
```