Chapter 10 - Canonical Correlation Analysis

- Quantify association between 2 sets of variables.
- Correlation between linear combinations of the 2 sets of variables. Start w/ lin. combos. w/ highest correlation, then followed by linear combinations that are uncorrelated w/ previous ones.
- Pairs of linear combinations ≡ Canonical variables
- Correlations ≡ Canonical Correlations.

10.2 Canonical Variates and Canonical Correlations

Random vectors: $X^{(1)} \equiv p\times 1 \qquad X^{(2)} \equiv q\times 1 \qquad p \le q$

$$E\{X^{(1)}\} = \mu^{(1)} \qquad V\{X^{(1)}\} = \Sigma_{11}$$

$$E\{X^{(2)}\} = \mu^{(2)} \qquad V\{X^{(2)}\} = \Sigma_{22}$$

$$Cov\{X^{(1)}, X^{(2)}\} = \Sigma_{12} = \Sigma_{21}'$$

$$\underset{(p+q)\times 1}{\underset{\sim}{X}} = \begin{bmatrix} \underset{\sim}{X}^{(1)} \\ \hdashline \underset{\sim}{X}^{(2)} \end{bmatrix} = \begin{bmatrix} X_1^{(1)} \\ \vdots \\ X_p^{(1)} \\ \hdashline X_1^{(2)} \\ \vdots \\ X_q^{(2)} \end{bmatrix} \qquad \underset{\sim}{\mu} = E\{\underset{\sim}{X}\} = \begin{bmatrix} E\{X^{(1)}\} \\ E\{X^{(2)}\} \end{bmatrix} = \begin{bmatrix} \mu^{(1)} \\ \hdashline \mu^{(2)} \end{bmatrix}$$

$$\Sigma = E\{(\underset{\sim}{X}-\underset{\sim}{\mu})(\underset{\sim}{X}-\underset{\sim}{\mu})'\} = \begin{matrix} E\{(X^{(1)}-\mu^{(1)})(X^{(1)}-\mu^{(1)})'\} & E\{(X^{(1)}-\mu^{(1)})(X^{(2)}-\mu^{(2)})'\} \\ E\{(X^{(2)}-\mu^{(2)})(X^{(1)}-\mu^{(1)})'\} & E\{(X^{(2)}-\mu^{(2)})(X^{(2)}-\mu^{(2)})'\} \end{matrix}$$

$$\Rightarrow \Sigma = \begin{bmatrix} \underset{p\times p}{\Sigma_{11}} & \underset{p\times q}{\Sigma_{12}} \\ \underset{q\times p}{\Sigma_{21}} & \underset{q\times q}{\Sigma_{22}} \end{bmatrix}$$

Goal: Summarize info. in Σ_{12} through a few linear combinations of the variables

$$U = \underset{\sim}{a}'\underset{\sim}{X}^{(1)} \qquad V = \underset{\sim}{b}'\underset{\sim}{X}^{(2)}$$

$$V\{U\} = \underset{\sim}{a}'V\{X^{(1)}\}\underset{\sim}{a} = \underset{\sim}{a}'\Sigma_{11}\underset{\sim}{a}$$

$$V\{V\} = \underset{\sim}{b}'V\{X^{(2)}\}\underset{\sim}{b} = \underset{\sim}{b}'\Sigma_{22}\underset{\sim}{b}$$

$$Cov\{U,V\} = \underset{\sim}{a}'Cov\{X^{(1)},X^{(2)}\}\underset{\sim}{b} = \underset{\sim}{a}'\Sigma_{12}\underset{\sim}{b}$$

$$Corr\{U,V\} = \frac{\underset{\sim}{a}'\Sigma_{12}\underset{\sim}{b}}{\sqrt{\underset{\sim}{a}'\Sigma_{11}\underset{\sim}{a}}\sqrt{\underset{\sim}{b}'\Sigma_{22}\underset{\sim}{b}}}$$

Want $\underset{\sim}{a}, \underset{\sim}{b}$ to make this correlation as large as possible.

$$\underset{\sim}{a}'\Sigma_{11}\underset{\sim}{a} = \underset{\sim}{b}'\Sigma_{22}\underset{\sim}{b} = 1$$

At each stage

U_k, V_k w/ unit variances, uncorrelated w/ previous linear combinations, that maximize correlation.

Result 10.1 $\quad p \le q \qquad X^{(1)}, X^{(2)}$

$Cov\{X^{(1)}\} = \Sigma_{11} \quad Cov\{X^{(2)}\} = \Sigma_{22} \quad Cov\{X^{(1)}, X^{(2)}\} = \Sigma_{12}$

where Σ = full rank

for fixed vectors $\underset{p\times 1}{\underset{\sim}{a}}, \underset{q\times 1}{\underset{\sim}{b}}$ form $U = \underset{\sim}{a}'\underset{\sim}{X}^{(1)}, \; V = \underset{\sim}{b}'\underset{\sim}{X}^{(2)}$

$$\max_{a,b} Corr\{U, V\} = \rho_1^*$$

attained by: $U_1 = \underset{\sim}{a}_1'X^{(1)} = \underset{\sim}{e}_1' \Sigma_{11}^{-1/2} \underset{\sim}{X}^{(1)}$

$V_1 = \underset{\sim}{b}_1'X^{(2)} = \underset{\sim}{f}_1' \Sigma_{22}^{-1} X^{(2)}$

k^{th} pair $(k = 2, \ldots, p)$

$U_k = \underset{\sim}{a}_k'X^{(1)} = \underset{\sim}{e}_k' \Sigma_{11}^{-1/2} X^{(1)} \qquad V_k = \underset{\sim}{b}_k'X^{(2)} = \underset{\sim}{f}_k' \Sigma_{22}^{-1} X^{(2)}$

maximizes ρ_k^* s.t. uncorrelated w/ preceding canonical variables.

$\rho_1^* \ge \rho_2^* \ge \ldots \ge \rho_p^* \equiv$ eigenvalues of $\Sigma_{11}^{-1/2} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-1/2}$

$\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_p \equiv$ corresponding $p\times 1$ eigenvectors.

$\underset{\sim}{f}_1, \ldots, \underset{\sim}{f}_p \equiv$ eigenvectors of of $\Sigma_{22}^{-1/2} \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1/2}$

Standardized Variables

$$Z^{(1)} = \begin{bmatrix} Z_1^{(1)} \\ \vdots \\ Z_p^{(1)} \end{bmatrix} \qquad Z^{(2)} = \begin{bmatrix} Z_1^{(2)} \\ \vdots \\ Z_q^{(2)} \end{bmatrix}$$

$$U_k = \underset{\sim}{a}_k' Z^{(1)} \qquad V_k = \underset{\sim}{b}_k' Z^{(2)} = \underset{\sim}{f}_k' \rho_{22}^{-1/2} Z^{(2)}$$

$$= \underset{\sim}{e}_k' \rho_{11}^{-1/2} Z^{(1)}$$

$$\mathrm{Cov}\{Z^{(1)}\} = \rho_{11} \qquad \mathrm{Cov}\{Z^{(2)}\} = \rho_{22} \qquad \mathrm{Cov}\{Z^{(1)}, Z^{(2)}\} = \rho_{12} = \rho_{21}'$$

$\underset{\sim}{e}_k \equiv$ eigenvectors of $\rho_{11}^{-1/2}\rho_{12}\rho_{22}^{-1}\rho_{21}\rho_{11}^{-1/2}$

$\underset{\sim}{f}_k \equiv$ " " $\rho_{22}^{-1/2}\rho_{21}\rho_{11}^{-1}\rho_{12}\rho_{22}^{-1/2}$

$\rho_1^{*2} \geq \dots \geq \rho_p^{*2} \equiv$ non-zero eigenvalues of $\rho_{11}^{-1/2}\rho_{12}\rho_{22}^{-1}\rho_{21}\rho_{11}^{-1/2}$

Note: If $\underset{\sim}{a}_k$ and $\underset{\sim}{b}_k$ are obtained from $X^{(1)}, X^{(2)}$

then $\underset{\sim}{a}_k' V_{11}^{1/2}$ and $\underset{\sim}{b}_k' V_{22}^{1/2}$ are " " $Z^{(1)}, Z^{(2)}$

where $V_{kk}^{1/2} \equiv$ diagonal matrix with $\sqrt{\text{variances}} = \sqrt{\sigma_{ii}}$

Alternative Approach to Computing $\underset{\sim}{a}_i, \underset{\sim}{b}_i$ from R

$$E_1 = R_{11}^{-1} R_{12} R_{22}^{-1} R_{21} \qquad E_2 = R_{22}^{-1} R_{21} R_{11}^{-1} R_{12}$$

$\underset{\sim}{a}_i \equiv$ eigenvectors of E_1 $\qquad$ $\underset{\sim}{b}_i \equiv$ eigenvectors of E_2

Canonical correlations $\equiv \sqrt{\ }$ of non-zero eigenvalues of E_1

10.3 Interpreting Population Canonical variables

$$\underset{p\times 1}{U} = A X^{(1)} = [\underset{\sim}{a}_1 \cdots a_p] X^{(1)}$$

$$\underset{q\times 1}{V} = B X^{(2)} = [b_1 \cdots b_q] X^{(2)}$$

$$\text{Cov}\{\underset{\sim}{U}, \underset{\sim}{X}_1) = \text{Cov}\{A\underset{\sim}{X}_1, \underset{\sim}{X}_1\} = A\Sigma_{11}$$

$$\text{Cor}\{\underset{\sim}{U}_1, \underset{\sim}{X}_1) = \text{Cov}\{\underset{\sim}{U}_1, V_{11}^{-1/2} X^{(1)}) = A\Sigma_{11} V_{11}^{-1/2}$$

$$\Rightarrow \underset{p\times p}{\rho_{U,X^{(1)}}} = A\Sigma_{11} V_{11}^{-1/2} \qquad \underset{q\times q}{\rho_{V,X^{(2)}}} = B\Sigma_{22} V_{22}^{-1/2}$$

$$\underset{p\times q}{\rho_{U,X^{(2)}}} = A\Sigma_{12} V_{22}^{-1/2} \qquad \underset{q\times p}{\rho_{V,X^{(1)}}} = B\Sigma_{21} V_{11}^{-1}$$

$$\rho_{U,Z^{(1)}} = A_Z \rho_{11} \qquad \rho_{V,Z^{(2)}} = B_Z \rho_{22}$$

$$\rho_{U,Z^{(2)}} = A_Z \rho_{12} \qquad \rho_{V,Z^{(1)}} = B_Z \rho_{21}$$

Note $\underset{\sim}{a_i}, \underline{b_i}$ chosen to maximize correlations, not necessarily explain variation in Σ_{11}, Σ_{22}. In this case, not sure how to interpret high correlation coefficients.

10.4 Sample Canonical Variates and Correlations

Random Sample of n observations on $X^{(1)}, X^{(2)}$:

$$\underset{n\times(p+q)}{X} = \begin{bmatrix} x_{11}^{(1)} & \cdots & x_{1p}^{(1)} & \vdots & x_{11}^{(2)} & \cdots & x_{1q}^{(2)} \\ \vdots & & \vdots & \vdots & \vdots & & \\ x_{n1}^{(1)} & & x_{np}^{(1)} & \vdots & x_{n1}^{(2)} & \cdots & x_{nq}^{(2)} \end{bmatrix} \qquad \underset{(p+q)\times 1}{\bar{X}} = \begin{bmatrix} \bar{x}^{(1)} \\ \cdots \\ \bar{x}^{(2)} \end{bmatrix}$$

$$S = \frac{1}{n-1} X'\left[I - \frac{1}{n}J\right]X = \left[\begin{array}{c|c} \underset{p\times p}{S_{11}} & \underset{p\times q}{S_{12}} \\ \hline \underset{q\times p}{S_{21}} & \underset{q\times q}{S_{22}} \end{array}\right]$$

Linear Combinations: $\hat{U} = \underline{\hat{a}}'\underline{X}^{(1)} \quad \hat{V} = \underline{\hat{b}}'x_2$

$$\Rightarrow \widehat{Corr}\{\hat{U}, \hat{V}\} = \frac{\hat{a}' S_{12} \hat{b}}{\sqrt{\hat{a}' S_{11} \hat{a}}\sqrt{\hat{b}' S_{11} \hat{b}}} = r_{\hat{U},\hat{V}}$$

1^{st} pair of sample canonical variates maximize $r_{\hat{U},\hat{V}}$

s.t. $\hat{a}' S_{11} \hat{a} = \hat{b}' S_{22} \hat{b} = 1$

⋮

k^{th} pair max. $r_{\hat{U},\hat{V}}$ s.t. they are uncorrelated with $k-1$ previous ones.

Result 10.2 $\hat{\rho}_1^{*2} \geq \cdots \geq \hat{\rho}_p^{*2} \equiv p$ ordered

eigenvalues of $S_{11}^{-1/2} S_{12} S_{22}^{-1} S_{21} S_{11}^{-1/2}$ w/ eigenvectors $\hat{\underset{\sim}{e}}_1, \ldots, \hat{\underset{\sim}{e}}_p$

$(p \leq q)$

$\hat{\underset{\sim}{f}}_1, \ldots, \hat{f}_p \equiv$ eigenvectors of $S_{22}^{-1/2} S_{21} S_{11}^{-1} S_{12} S_{22}^{-1/2}$

(1st p can also be obtained from $\hat{f}_k = \left(\frac{1}{\hat{\rho}_k^*}\right) S_{22}^{-1/2} S_{21} S_{11}^{-1/2} \hat{\underset{\sim}{e}}_k$)

$k = 1, \ldots, p$

$\hat{U}_k = \hat{\underset{\sim}{e}}_k' S_{11}^{-1/2} \underset{\sim}{X}^{(1)} = a_k' \underset{\sim}{x}^{(1)}$ for a given unit (observation)

$\hat{V}_k = \hat{f}_k' S_{22}^{-1/2} \underset{\sim}{X}^{(2)} = \underset{\sim}{b}_k' \underset{\sim}{X}^{(2)}$ for given unit.

$\hat{A} = [\hat{\underset{\sim}{a}}_1 \ \ldots \ \hat{\underset{\sim}{a}}_p]'$ $\quad \hat{B} = [\hat{\underset{\sim}{b}}_1 \cdots \hat{\underset{\sim}{b}}_q]'$

(Last $q-p$ eigenvalues are zero, but have mutually orthogonal eigenvalues).

$\underset{p\times 1}{\hat{U}} = \hat{A} X^{(1)}$ $\quad \underset{q \times 1}{\hat{V}} = \hat{B} x^{(2)}$

$R_{\hat{U}, x^{(1)}} = \hat{A} S_{11} D_{11}^{-1/2}$ $\quad R_{\hat{V}, x^{(2)}} = \hat{B} S_{22} D_{22}^{-1/2}$

D_{11} = Diagonal matrix w/ $\left\{ \left[S_{ii}^{(1)}\right]^{-1/2} \right\}$

$R_{\hat{U}, x^{(2)}} = \hat{A} S_{12} D_{22}^{-1/2}$ $\quad R_{\hat{V}, x^{(1)}} = \hat{B} S_{21} D_{11}^{-1/2}$

Standardized observations

$$Z = \left[\underset{n\times p}{Z^{(1)}} \,\vdots\, \underset{n\times q}{Z^{(2)}} \right] = \begin{bmatrix} z_1^{(1)\prime} & \vdots & z_1^{(2)\prime} \\ \vdots & & \vdots \\ z_n^{(1)\prime} & \vdots & z_n^{(2)\prime} \end{bmatrix}$$

$$\underset{p\times 1}{\hat{U}} = \hat{A}_z z^{(1)} \qquad \underset{q\times 1}{\hat{V}} = \hat{B}_z z^{(2)}$$

$$\hat{A}_z = \hat{A} D_{11}^{1/2} \qquad \hat{B}_z = \hat{B} D_{22}^{1/2}$$

To obtain Correlations between $\hat{U}, z^{(1)}, z^{(2)}, \hat{V}, z^{(1)}, z^{(2)}$

use previous formulas w/ $\hat{A}_z$ for $\hat{A}$, $\hat{B}_z$ for $\hat{B}$, R for S

10.5 Sample Descriptive Measures

Matrices of Error Approximations

$\hat{A} = [\hat{\underset{\sim}{a}}_1 \cdots \hat{\underset{\sim}{a}}_p]'$ $\qquad \hat{B} = [\hat{\underset{\sim}{b}}_1 \ldots \hat{\underset{\sim}{b}}_q]'$

$\underset{p\times 1}{\hat{U}} = \hat{A}\underline{x}^{(1)}$ $\qquad \underset{q\times 1}{\hat{V}} = \hat{B}\underline{x}^{(2)}$

~~Let $\hat{A}^{-1} =$~~

$$\hat{A} = \begin{bmatrix} \hat{a}_1' \\ \vdots \\ \hat{a}_p' \end{bmatrix} \qquad \hat{A}\hat{A}^{-1} = \begin{bmatrix} \hat{\underset{\sim}{a}}_1' \\ \vdots \\ \hat{\underset{\sim}{a}}_p \end{bmatrix} \begin{bmatrix} \hat{a}^{(1)} \vdots & \cdots & \hat{a}^{(p)} \end{bmatrix} = I$$

$$\Rightarrow \underset{\sim}{x}^{(1)} = \hat{A}^{-1}\underset{\sim}{\hat{U}} \qquad \underline{X}^{(2)} = \hat{B}^{-1}\underline{\hat{V}}$$

$$Cov\{\hat{U},\hat{V}\} = \hat{A}S_{12}\hat{B}' \qquad Cov\{\hat{U}\} = \hat{A}S_{11}\hat{A}' = I_p$$

$$Cov\{\hat{V}\} = \hat{B}S_{22}\hat{B}' = I_q$$

$$Cov\{\hat{U},\hat{V}\} = \left[\begin{array}{ccc|c} \hat{\ell}_1^* & & 0 & \\ & \ddots & & 0 \\ 0 & & \hat{\ell}_p^* & \end{array}\right]$$

$$\Rightarrow S_{12} = \hat{A}^{-1}\left[\begin{array}{ccc|c} \hat{\ell}_1^* & & 0 & \\ 0 & \ddots & \hat{\ell}_p^* & 0 \end{array}\right](\hat{B}^{-1})'$$

$$S_{11} = \hat{A}^{-1}(\hat{A}^{-1})' \qquad S_{22} = \hat{B}^{-1}(\hat{B}^{-1})'$$

$$\Rightarrow S_{12} = \hat{\ell}_1^*\hat{a}^{(1)}\hat{b}^{(1)'} + \cdots + \hat{\ell}_p^*\hat{a}^{(p)}\hat{b}^{(p)'}$$

$$S_{11} = \hat{a}^{(1)}\hat{a}^{(1)'} + \cdots + \hat{a}^{(p)}\hat{a}^{(p)'}$$

$$S_{22} = \hat{b}^{(1)}\hat{b}^{(1)'} + \cdots + \hat{b}^{(q)}\hat{b}^{(q)'}$$

when using first r canonical pairs:

$$\tilde{x}^{(1)} = [\hat{a}^{(1)} \dots \hat{a}^{(r)}] \begin{bmatrix} \hat{U}_1 \\ \vdots \\ \hat{U}_r \end{bmatrix} \qquad \tilde{x}^{(2)} = [\hat{b}^{(1)} \dots \hat{b}^{(r)}] \begin{bmatrix} \hat{V}_1 \\ \vdots \\ \hat{V}_r \end{bmatrix}$$

S_{12} is approximated by $Cov\{\tilde{x}^{(1)}, \tilde{x}^{(2)}\}$

$$S_{11} - (\hat{a}^{(1)}\hat{a}^{(1)\prime} + \dots + \hat{a}^{(r)}\hat{a}^{(r)\prime}) = \hat{a}^{(r+1)}\hat{a}^{(r+1)\prime} + \dots + \hat{a}^{(p)}\hat{a}^{(p)\prime}$$

$$S_{22} - (\hat{b}^{(1)}\hat{b}^{(1)\prime} + \dots + \hat{b}^{(r)}\hat{b}^{(r)\prime}) = \hat{b}^{(r+1)}\hat{b}^{(r+1)\prime} + \dots + \hat{b}^{(q)}\hat{b}^{(q)\prime}$$

$$S_{12} - (\hat{\rho}_1^* \hat{a}^{(1)}\hat{b}^{(1)\prime} + \dots + \hat{\rho}_r^* \hat{a}^{(r)}\hat{b}^{(r)\prime}) = \hat{\rho}_{r+1}\hat{a}^{(r+1)}\hat{b}^{(r+1)\prime} + \dots + \hat{\rho}_p \hat{a}^{(p)}\hat{b}^{(p)\prime}$$

Similar results for R_{kl} w/ $\hat{a}_z^{(k)}, \hat{b}_z^{(l)}$.

Errors tend to be smaller for S_{12} as $\hat{\rho}_{r+1}, \dots, \hat{\rho}_p$ tend to be small due to nature of process

Proportions of explained sample variance

$$\hat{Cov}\{z^{(1)}, \hat{U}\} = \hat{Cov}\{\hat{A}_z^{-1}\hat{U}, \hat{U}\} = \hat{A}_z^{-1}$$

$$\hat{Cov}\{z^{(2)}, \hat{V}\} = \hat{B}_z^{-1}$$

$$\hat{A}_z^{-1} = \begin{bmatrix} r_{\hat{U}_1, z_1^{(1)}} & \cdots & r_{\hat{U}_p, z_1^{(1)}} \\ \vdots & & \\ r_{\hat{U}_1, z_p^{(1)}} & \cdots & r_{\hat{U}_p, z_p^{(1)}} \end{bmatrix} \qquad \hat{B}_z^{-1} = \begin{bmatrix} r_{\hat{V}_1, z_1^{(2)}} & \cdots & r_{\hat{V}_q, z_1^{(2)}} \\ \vdots & \ddots & \vdots \\ r_{\hat{V}_1, z_q^{(2)}} & \cdots & r_{\hat{V}_q, z_q^{(2)}} \end{bmatrix}$$

$$R_{11} = \hat{A}_z^{-1}\left(\hat{A}_z^{-1}\right)' \Rightarrow tr(R_{11}) = tr\left\{\hat{a}_z^{(1)}\hat{a}_z^{(1)'} + \cdots + \hat{a}_z^{(p)}\hat{a}_z^{(p)'}\right\} = p$$

$$tr(R_{22}) = tr\left\{\hat{b}_z^{(1)}\hat{b}_z^{(1)'} + \cdots + \hat{b}_z^{(q)}\hat{b}_z^{(q)'}\right\} = q$$

Contributions of first r canonical variates

$$tr\left(\hat{a}_z^{(1)}\hat{a}_z^{(1)'} + \cdots + \hat{a}_z^{(r)}\hat{a}_z^{(r)'}\right) = \sum_{i=1}^{r}\sum_{k=1}^{p} r^2_{\hat{U}_i, z_k^{(1)}}$$

and

$$tr\left(\hat{b}_z^{(1)}\hat{b}_z^{(1)'} + \cdots + \hat{b}_z^{(r)}\hat{b}_z^{(r)'}\right) = \sum_{i=1}^{r}\sum_{k=1}^{q} r^2_{\hat{V}_i, z_k^{(2)}}$$

Proportions: $$R^2_{z^{(1)}|\hat{U}_1,\ldots,\hat{U}_r} = \frac{tr\left(\hat{a}_z^{(1)}\hat{a}_z^{(1)'} + \cdots + \hat{a}_z^{(r)}\hat{a}_z^{(r)'}\right)}{p}$$

$$R^2_{z^{(2)}|\hat{V}_1,\ldots,\hat{V}_r} = \frac{tr\left(\hat{b}_z^{(1)}\hat{b}_z^{(1)'} + \cdots + \hat{b}_z^{(r)}\hat{b}_z^{(r)'}\right)}{q}$$

10.6 Large-Sample Inferences

$$X_j = \begin{bmatrix} X_j^{(1)} \\ X_j^{(2)} \end{bmatrix} \equiv \text{Radom Sample} \sim N_{p+q}(\mu, \Sigma)$$

$$\Sigma = \begin{bmatrix} \underset{p\times p}{\Sigma_{11}} & \underset{p\times q}{\Sigma_{12}} \\ \underset{q\times p}{\Sigma_{21}} & \underset{q\times q}{\Sigma_{22}} \end{bmatrix}$$

$H_0: \Sigma_{12} = 0 \qquad H_A: \Sigma_{12} \neq 0$

$$-2 \ln \Lambda = n \ln\left(\frac{|S_{11}|\,|S_{22}|}{|S|}\right) = -n \ln \prod_{i=1}^{p} (1 - \hat{\rho}_i^{*2})$$

where $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ is unbiased estimator for Σ

Note: $\Sigma_{12} = 0 \Rightarrow \rho_1^* = \cdots = \rho_p^* = 0$

Bartlett Correction $\quad -\left(n - 1 - \frac{p+q+1}{2}\right) \ln \prod_{i=1}^{p} (1 - \hat{\rho}_i^{*2}) > \chi^2_{pq}(\alpha)$

Testing a sequence of (ordered) ρ^*s is zero if H_0 rejected

$H_0^k: \rho_1^* \neq 0, \ldots, \rho_k^* \neq 0, \rho_{k+1}^2 = \cdots = \rho_p^2 = 0$

$H_A^k: \rho_i^* \neq 0$ for some $i \geq k+1$

Reject H_0^k if

$$-\left(n - 1 - \frac{p+q+1}{2}\right) \ln \prod_{i=k+1}^{p} (1 - \hat{\rho}_i^{*2}) \geq \chi^2_{(p-k)(q-k)}(\alpha)$$