STA 4321/5325 – Exam 3 – Fall 2015

/115 TOTAL

PRINT Name Answer Key **UFID** ________

SHOW ALL WORK!!!!!!!!!!!!!!!!!!!!!!!!!

For $a>0,\ b>0$: $\int_0^\infty y^{a-1}e^{-y/b}dy=\Gamma(a)b^a$	$\int_0^1 y^{a-1}(1-y)^{b-1}dy=\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$	$\Gamma(a)=(a-1)\Gamma(a-1)$

Q.1. Derive the moment-generating function for the Exponential distribution with parameter β.

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$$f(y)=\frac{1}{\beta}e^{-y/\beta}\qquad y\geq 0,\ \beta>0$$

$$m(t)=E\{e^{tY}\}=\int_0^\infty e^{ty}\frac{1}{\beta}e^{-y/\beta}dy$$

$$=\int_0^\infty \frac{1}{\beta}\exp\left\{-y\left(-t+\frac{1}{\beta}\right)\right\}dy$$

$$=\int_0^\infty \frac{1}{\beta}\exp\left\{-y\left(\frac{1-\beta t}{\beta}\right)\right\}dy$$

$$=\frac{1}{\beta}\int_0^\infty \exp\left\{-\frac{y}{(\beta/(1-\beta t))}\right\}dy$$

$$=\frac{1}{\beta}\int_0^\infty \exp\left\{-\frac{y}{\beta^*}\right\}dy\qquad \beta^*=\frac{\beta}{1-\beta t}$$

$$=\frac{1}{\beta}\Gamma(1)(\beta^*)^1=\frac{1}{\beta}\left(\frac{\beta}{1-\beta t}\right)=(1-\beta t)^{-1}$$

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Q.2. The proportion of a particular pigment in batches of fabric dye is a random variable that is distributed as Beta. That is, $f(y)=\begin{cases}\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}(1-y)^{\beta-1} & 0\le y\le 1 \quad \alpha,\beta>0\\ 0 & \text{elsewhere}\end{cases}$

p.2.a. **Derive** the mean of Y, the proportion of pigment in a randomly selected batch.

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$$E\{Y\} = \int_0^1 y \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}\,dy$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{(\alpha+1)-1}(1-y)^{\beta-1}\,dy$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \frac{\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)} \frac{\alpha\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{\alpha}{\alpha+\beta}$$

p.2.b. What is the probability that the proportion of the pigment in a randomly selected batch is over 0.5, if α=1, and β=2.

8

$$\int_{.5}^1 \frac{\Gamma(3)}{\Gamma(1)\Gamma(2)} y^{1-1}(1-y)^{2-1}\,dy$$

$$= 2\int_{.5}^1 (1-y)\,dy = 2\left[y - \frac{y^2}{2}\Big|_{.5}^1 \right]$$

$$= 2\left[\left(1-\frac{1}{2}\right) - \left(\frac{1}{2}-\frac{1}{8}\right)\right] = 2\left(\frac{1}{8}\right) = \frac{1}{4}$$

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Q.3. Two food reviewers have the following joint probability distribution of ratings across a wide variety of food offerings. Negative reviews are assigned -1, mixed reviews 0, and positive reviews 1. Let Y_1 be reviewer A's review and Y_2 be reviewer B's review for a randomly selected item.

Y_2

Reviewer A\B	-1	0	1
-1	0.15	0.10	0.05
0	0.10	0.20	0.10
1	0.05	0.05	0.20

Y_1

p.3.a. Obtain the marginal probability distributions for Y_1 and Y_2.

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y_1	$P_1(y_1)$
-1	.30
0	.40
$+1$	.30

y_2	$P(y_2)$
-1	.30
0	.35
$+1$	.35

p.3.b. Given that Y_1=0, give the conditional distribution for Y_2.

$$P(Y_2 \mid Y_1 = 0) = \frac{P(0, y_2)}{P_1(0)}$$

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y_2	$P(y_2 \mid y_1 = 0)$
-1	$.10/.40 = .25$
0	$.20/.40 = .50$
1	$.10/.40 = .25$

Reversed

y_1	$P(y_1 \mid y_2 = 0)$
-1	$.10/.35 = 2/7$
0	$.20/.35 = 4/7$
1	$.05/.35 = 1/7$

p.3.c. Compute the covariance of Y_1 and Y_2, where:

$$COV\{Y_1, Y_2\} = E\{(Y_1 - E\{Y_1\})(Y_2 - E\{Y_2\})\} = E\{Y_1 Y_2\} - E\{Y_1\}E\{Y_2\}$$

8

$$E\{Y_1 Y_2\} = (-1)(-1)(.15) + (-1)(1)(.05) + 1(-1)(.05) + 1(1)(.20) + 0$$

$$= .15 - .05 - .05 + .20 = .25$$

$$E\{Y_1\} = 0 \quad E\{Y_2\} = .05$$

$$\Rightarrow Cov\{Y_1, Y_2\} = .25 - 0(.05) = .25$$

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Q.4. A random variable Y has moment-generating function:

$$m(t) = e^{\left\{100t + \frac{400t^2}{2}\right\}}$$

p.4.a. Use this to **derive** $E\{Y\}$, $E\{Y^2\}$, and $V\{Y\}$

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$$m'(t) = \left(100 + \frac{800t}{2}\right) \exp\left\{100t + \frac{400t^2}{2}\right\}$$

$$m''(t) = \frac{800}{2} \exp\left\{100t + \frac{400t^2}{2}\right\} + \left(100 + \frac{800t}{2}\right)^2 \exp\left\{100t + \frac{400t^2}{2}\right\}$$

$$m'(0) = E\{Y\} = (100+0)e^0 = 100$$

$$m''(0) = E\{Y^2\} = 400e^0 + (100+0)^2 e^0$$

$$= 400 + 100^2$$

$$V\{Y\} = E\{Y^2\} - \left(E\{Y\}\right)^2 = 400 + 100^2 - 100^2 = 400$$

p.4.b. Specifically, what is the probability distribution of Y (name the family and parameters).

3

$$Y \sim N(\mu = 100, \sigma^2 = 400)$$

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Q.5. Y_1 and Y_2 denote random lifetimes of two types (type I and type II) of electronic components. The joint density function is given by:

$$f(y_1, y_2) = \begin{cases} (1/8)\, y_1 e^{-(y_1+y_2)/2} & y_1 > 0, y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

p.5.a. Obtain the marginal probability distribution for Y_1 (be specific of ranges)

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$$f_1(y_1) = \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2}\, dy_2$$

$$= \frac{1}{\cancel{8}4} y_1 e^{-y_1/2} \underbrace{\int_0^\infty \frac{1}{2} e^{-y_2/2}\, dy_2}_{\text{EXP(2) Density}} = \begin{cases} \frac{1}{4} y_1 e^{-y_1/2} & y_1 > 0 \\ 0 & \text{e.w.} \end{cases}$$

p.5.b. Obtain the marginal probability distribution for Y_2 (be specific of ranges)

8

$$f_2(y_2) = \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2}\, dy_1 = \frac{1}{2} e^{-y_2} \int_0^\infty \frac{1}{4} y_1 e^{-y_1/2}\, dy_1$$

Note: $\alpha = 2,\ \beta = 2 \quad \Gamma(\alpha) = 1 \quad \beta^\alpha = 4 \Rightarrow \frac{1}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{4}$

$$\Rightarrow f_2(y_2) = \begin{cases} \frac{1}{2} e^{-y_2} & y_2 > 0 \\ 0 & \text{e.w.} \end{cases}$$

p.5.c. Give the conditional probability distribution for Y_2, given $Y_1 = y_1$

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$$f(y_2 | y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} = \frac{\frac{1}{8} y_1 e^{-(y_1+y_2)/2}}{\frac{1}{4} y_1 e^{-y_1/2}}$$

$$= \begin{cases} \frac{1}{2} e^{-y_2/2} & y_2 > 0 \\ 0 & \text{e.w.} \end{cases}$$

3 p.5.d. Are Y_1 and Y_2 independent? **Yes** **No**

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Q.5. Y_1 and Y_2 denote random lifetimes of two types (type I and type II) of electronic components. The joint density function is given by:

$$f(y_1, y_2) = \begin{cases} (1/8)\, y_1 e^{-(y_1+y_2)/2} & y_1 > 0, y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

p.5.a. Obtain the marginal probability distribution for Y_1 (be specific of ranges)

$$f_1(y_1) = \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_2 = \frac{1}{8} y_1 e^{-y_1/2} \int_0^\infty e^{-y_2/2} dy_2$$

$$= \frac{1}{8} y_1 e^{-y_1/2} \left[-\frac{1}{1/2} e^{-y_2/2} \Big|_0^\infty \right]$$

$$= \frac{1}{8} y_1 e^{-y_1/2} \left[-2 \left(e^{-\infty} - e^{-0} \right) \right] = \frac{1}{8} y_1 e^{-y_1/2} (2)$$

$$= \frac{1}{4} y_1 e^{-y_1/2} \quad y_1 > 0$$

p.5.b. Obtain the marginal probability distribution for Y_2 (be specific of ranges)

$$f_2(y_2) = \frac{1}{8} e^{-y_2/2} \int_0^\infty y_1 e^{-y_1/2} dy_1 \qquad u = y_1 \quad dv = e^{-y_1/2} dy_1$$

$$du = dy_1 \quad v = -2e^{-y_1/2}$$

$$\Rightarrow f_2(y_2) = \frac{1}{8} e^{-y_2/2} \left[-2y_1 e^{-y_1/2} \Big|_0^\infty - \int_0^\infty -2e^{-y_1/2} dy_1 \right]$$

$$= \frac{1}{8} e^{-y_2/2} \left\{ [0 - -0] + \frac{2}{-1/2} e^{-y_1/2} \Big|_0^\infty \right\}$$

$$= \frac{1}{8} e^{-y_2/2} \left\{ 0 + (0 - -4) \right\} = \frac{1}{2} e^{-y_2/2} \quad y_2 > 0$$

p.5.c. Give the conditional probability distribution for Y_2, given $Y_1 = y_1$

p.5.d. Are Y_1 and Y_2 independent? **Yes** **No**

Q.6. Times to complete a task (Y) for a population of students is exponential with mean $\beta = 2$. (Gamma(1,2))

p.6.a. Derive $F(y) = P(Y \le y)$

8

$$f(y) = \begin{cases} \frac{1}{2} e^{-y/2} & y \ge 0 \\ 0 & e.w. \end{cases}$$

$$F(y) = \frac{1}{2} \int_0^y e^{-t/2} dt = \frac{1}{2}\left(-\frac{1}{1/2} e^{-t/2}\Big|_0^y\right)$$

$$= -e^{-t/2}\Big|_0^y = 1 - e^{-y/2} \quad 0 \le y$$

p.6.b. What is the probability a random student takes more than 2 hours to complete the task.

4

$$P(Y>2) = 1 - F(2) = 1 - (1 - e^{-2/2}) = e^{-1} = 0.368$$

p.6.c. What is the probability a random student takes more than 4 hours to complete the task.

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$$P(Y>4) = 1 - F(4) = 1 - (1 - e^{-4/2}) = e^{-2} = 0.135$$

p.6.c. What is the probability a student takes more than 4 hours, given it takes more than 2 hours?

4

$$P(Y>4 \mid Y>2) = \frac{P(Y>4, Y>2)}{P(Y>2)}$$

$$= \frac{P(Y>4)}{P(Y>2)} = \frac{.135}{.368} = e^{-1} = .368$$

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