Likelihood Ratio, Wald, and Lagrange Multiplier (Score) Tests

Soccer Goals in European Premier Leagues - 2004

Statistical Testing Principles

- Goal: Test a Hypothesis concerning parameter value(s) in a larger population (or nature), based on observed sample data
- Data – Identified with respect to a (possibly hypothesized) probability distribution that is indexed by one or more unknown parameters
- Notation:

Data: $y_1, \ldots, y_n$

Parameter(s): $\theta_1, \ldots, \theta_k$

Joint Density Function: $f\left(y_1, \ldots, y_n \middle| \theta_1, \ldots, \theta_k\right)$

Example – English League – Total Goals/Match

- Suppose we wish to test whether the mean number of goals (in a hypothetically infinite population) of games is equal to 3. Note: all games of equal length (no overtime in regular season games)
- Data: Y=Total # of goals in a randomly selected game
- Distribution: Assume Poisson with parameter θ
- Null Hypothesis: H_0: $\theta = 3$
- Alternative Hypothesis: H_A: $\theta \neq 3$
- Joint Probability Density Function:

$$f\left(y_1,\ldots,y_n \middle| \theta\right) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{y_i}}{y_i!} = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!} \qquad y_i = 0,1,2,\ldots$$

Likelihood Function

- Another term for joint probability density/mass function. Common Notation: $L(\theta)$ or $L(\theta,y)$ or $L(\theta|y)$
- Considered as a function of both the (observed) data and the (unknown) parameter values
- Used in estimation and testing parameter value(s)
- Goal is to choose parameter value(s) that maximize likelihood function given the observed data.
- Typically work with the log of the likelihood, as it is often easier to differentiate to solve for maximum likelihood (ML) estimators for many families of probability distributions

ML Estimation of Poisson Mean

$$L(\theta, y) = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}$$

$$l = \ln\left(L(\theta, y)\right) = -n\theta + \left(\sum_{i=1}^{n} y_i\right)\ln(\theta) - \ln\left(\prod_{i=1}^{n} y_i!\right)$$

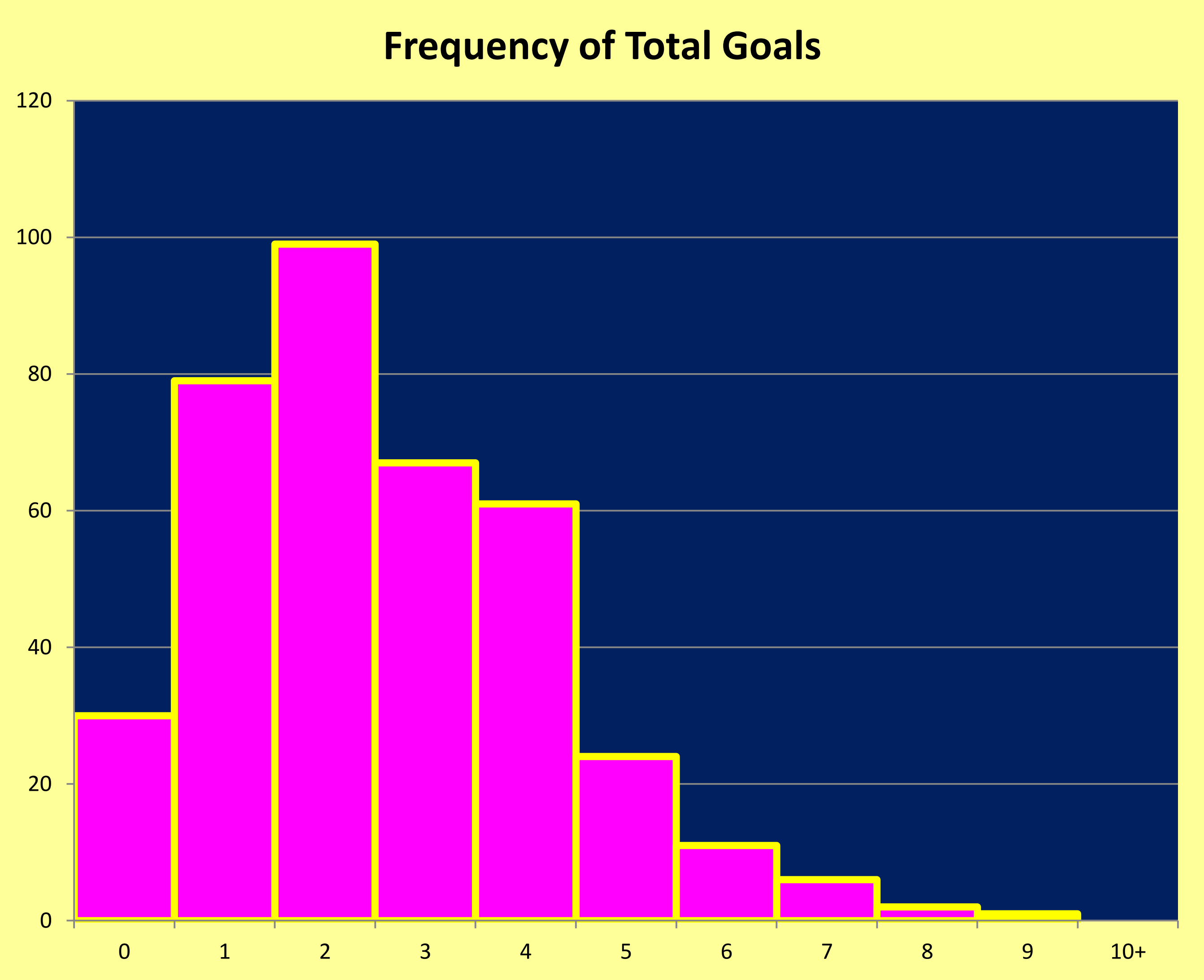
Taking derivative (wrt θ) and setting to zero for maximum:

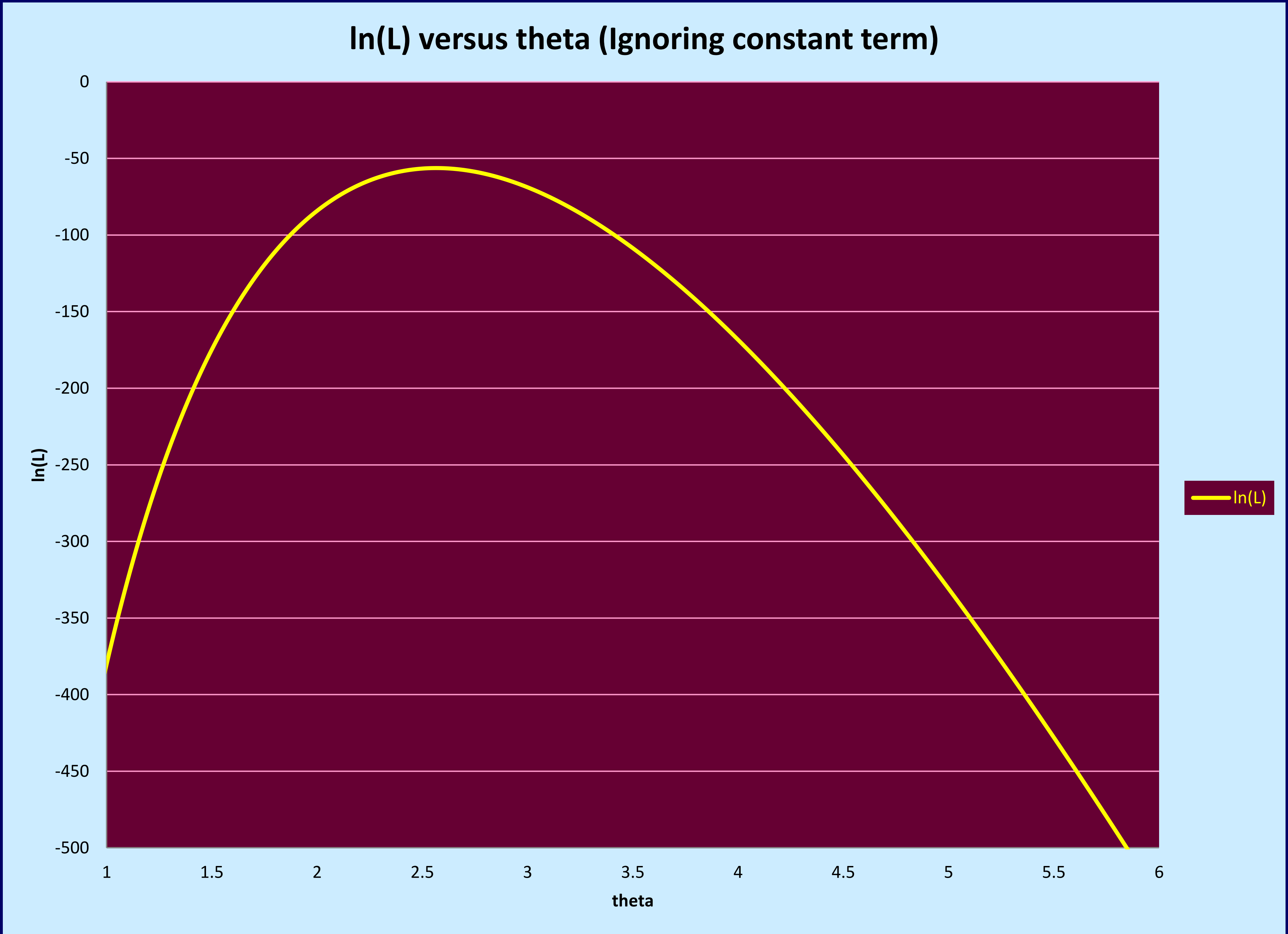
$$\frac{dl}{d\theta} = -n + \frac{\sum_{i=1}^{n} y_i}{\theta} - 0 \overset{set}{=} 0 \quad \Rightarrow \quad -n + \frac{\sum_{i=1}^{n} y_i}{\hat{\theta}} = 0 \quad \Rightarrow \quad \hat{\theta} = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{y}$$

Total Goals Data

Goals	Frequency
0	30
1	79
2	99
3	67
4	61
5	24
6	11
7	6
8	2
9	1
10+	0
Total	380

$$\hat{\theta} = \frac{\sum_{i=1}^{380} y_i}{380} = \frac{975}{380} = 2.57$$





ln(L) versus theta (Ignoring constant term)
0
-50
-100
-150
-200
-250
-300
-350
-400
-450
-500
ln(L)
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
theta
ln(L)

Likelihood Ratio Test

- Identify the parameter space: $\Omega = \{\theta : \theta > 0\}$
- Identify the parameter space under H_0: $\Omega_0 = \{\theta : \theta = \theta_0\}$
- Evaluate the maximum log-Likelihood
- Evaluate the log-Likelihood under H_0
- Any terms not involving parameter can be ignored
- Take -2 times difference (H_0 – maximum)
- Under null hypothesis (and large samples), statistic is approximately chi-square with 1 degree of freedom (number of constraints under H_0)

$$X_{LR}^2 = -2\left[\ln\left(L\left(\theta_0, y\right)\right) - \ln\left(L\left(\hat{\theta}, y\right)\right)\right]$$

Soccer Goals Example

$$\ln\left(L(\theta, y)\right) = -380\theta + \left(\sum_{i=1}^{380} y_i\right)\ln(\theta) - \ln\left(\prod_{i=1}^{380} y_i!\right)$$

Under $H_0 : \theta = 3$ (Ignoring $\ln\left(\prod_{i=1}^{380} y_i!\right)$):

$$\ln\left(L(\theta = 3, y)\right) = -380(3) + 975\left(\ln(3)\right) = -1140 + 1071.15 = -68.85$$

Maximum Value @ $\hat{\theta} = 2.57$:

$$\ln\left(L\left(\hat{\theta}, y\right)\right) = -380(2.57) + 975\left(\ln(2.57)\right) = -976.6 + 920.31 = -56.29$$

Test Statistic:

$$X_{LR}^2 = -2\left[\ln\left(L(\theta = 3, y)\right) - \ln\left(L\left(\hat{\theta}, y\right)\right)\right] = -2\left(-68.85 - (-56.29)\right) = 25.12 > \chi_{.05,1}^2 = 3.84$$

We have strong evidence to conclude the “true” mean total number of goals is below 3.

Wald Test - I

- By Central Limit Theorem arguments, many estimators have sampling distributions that are approximately normal in large samples
- Then, if we have an estimate of the variance of the estimator, we can obtain a chi-square statistic by taking the square of the distance between the ML estimate and the value under H_0 divided by the estimated variance
- The estimated variance can be obtained from the second derivative of the log-Likelihood

Wald Test - II

$$V\left(\hat{\theta}\right)=\frac{1}{n}I^{-1}(\theta) \quad \text{where: } I(\theta)=-\frac{1}{n}E\left[\frac{\partial^2 \ln(L)}{\partial\theta^2}\right]$$

Wald Chi-Square Statistic: $X_W^2=\frac{\left(\hat{\theta}-\theta_0\right)^2}{\hat{V}\left(\hat{\theta}\right)}=nI\left(\hat{\theta}\right)\left(\hat{\theta}-\theta_0\right)^2$

Poisson Model: $\ln\left(L(\theta,y)\right)=-n\theta+\left(\sum_{i=1}^{n}y_i\right)\ln(\theta)-\ln\left(\prod_{i=1}^{n}y_i!\right)$

$$\Rightarrow \quad \frac{\partial \ln\left(L(\theta,y)\right)}{\partial\theta}=-n+\frac{\sum_{i=1}^{n}y_i}{\theta} \quad \Rightarrow \quad \frac{\partial^2 \ln\left(L(\theta,y)\right)}{\partial\theta^2}=-\frac{\sum_{i=1}^{n}y_i}{\theta^2}$$

$$\Rightarrow \quad I(\theta)=-\frac{1}{n}E\left[\frac{\partial^2\ln(L)}{\partial\theta^2}\right]=-\frac{1}{n}\left(-\frac{n\theta}{\theta^2}\right)=\frac{1}{\theta}$$

$$\Rightarrow X_W^2=\frac{\left(\hat{\theta}-\theta_0\right)^2}{\hat{V}\left(\hat{\theta}\right)}=nI\left(\hat{\theta}\right)\left(\hat{\theta}-\theta_0\right)^2=\frac{n\left(\hat{\theta}-\theta_0\right)^2}{\hat{\theta}} = \frac{380\left(2.57-3\right)^2}{2.57}=27.34$$

Lagrange Multiplier (Score) Test

- Obtain the first derivative of the log-Likelihood evaluated at the parameter under H_0 (This is the slope of the log-Likelihood, evaluated at θ_0 and is called the **score**)
- Multiply the square of the score by the variance of the ML estimate, evaluated at θ_0. This is the inverse of the variance of the score.
- Then chi-square test statistic is computed as follows:

$$X_{LM}^2 = \frac{s(\theta_0, y)^2}{nI(\theta_0)} \quad \text{where} \;\; s(\theta, y) = \frac{\partial \ln\left(L(\theta, y)\right)}{\partial \theta}$$

Soccer Goals Example

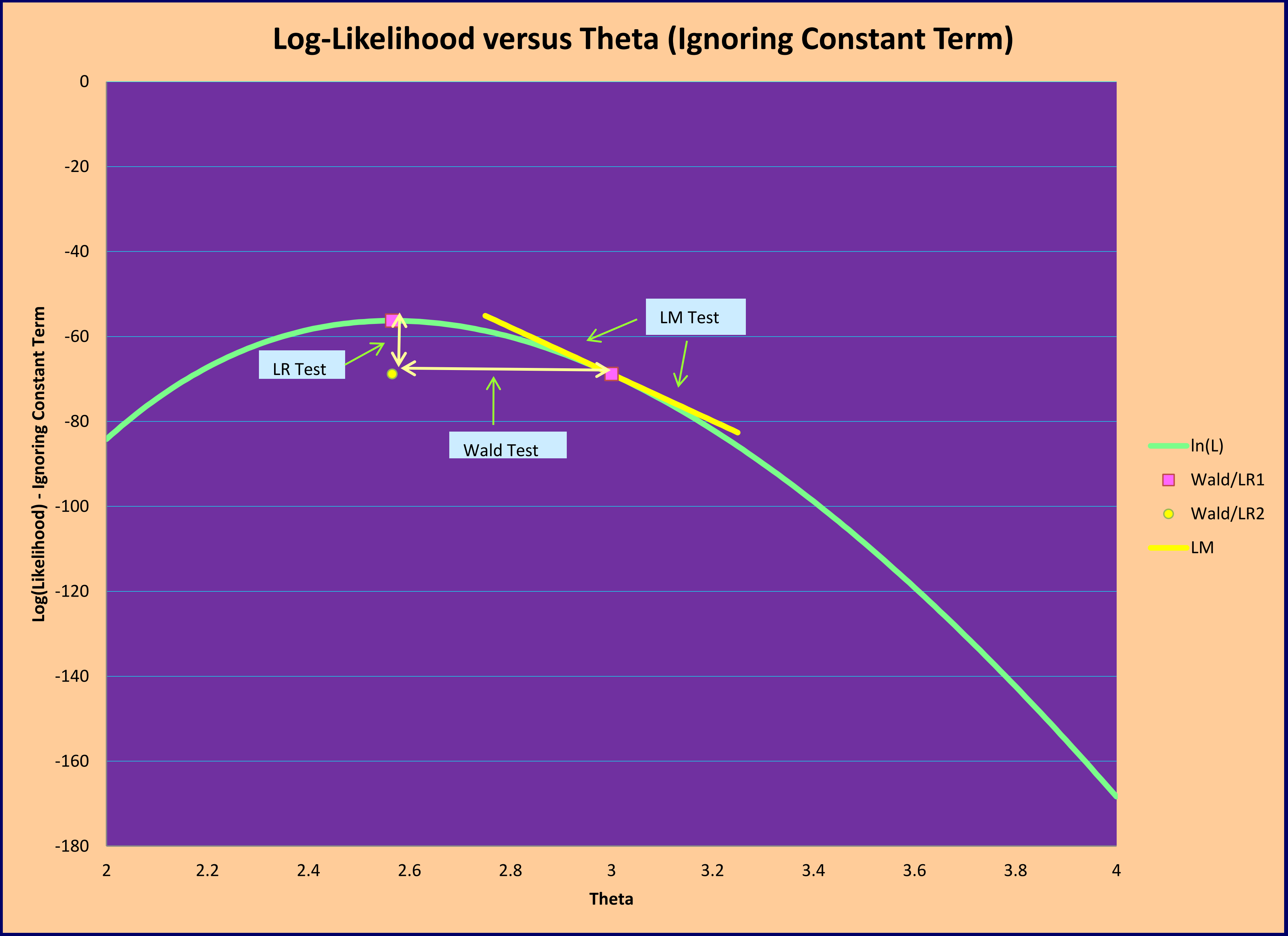
$$\ln\left(L(\theta, y)\right) = -n\theta + \left(\sum_{i=1}^{n} y_i\right)\ln(\theta) - \ln\left(\prod_{i=1}^{n} y_i!\right)$$

$$\Rightarrow \quad s(\theta, y) = \frac{\partial \ln\left(L(\theta, y)\right)}{\partial \theta} = -n + \frac{\sum_{i=1}^{n} y_i}{\theta} \Rightarrow \quad s(\theta_0, y) = -380 + \frac{975}{3} = -55$$

$$I(\theta) = \frac{1}{\theta} \quad \Rightarrow \quad I(\theta_0) = \frac{1}{\theta_0} = \frac{1}{3}$$

$$\Rightarrow \quad X_{LM}^2 = \frac{\left(s(\theta_0, y)\right)^2}{nI(\theta_0)} = \frac{(-55)^2}{385(1/3)} = 23.57$$

Note that: $X_W^2 = 27.34 > X_{LR}^2 = 25.12 > X_{LM}^2 = 23.57$


Log-Likelihood versus Theta (Ignoring Constant Term)
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
Log(Likelihood) - Ignoring Constant Term
LM Test
LR Test
Wald Test
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
3.8
4
Theta
ln(L)
Wald/LR1
Wald/LR2
LM

Generalization to Tests of Multiple Parameters

Parameter Vector: $\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_k \end{bmatrix}$ $\quad H_0 : R\theta = r \quad R = \begin{bmatrix} R_{11} & \cdots & R_{1k} \\ \vdots & \ddots & \vdots \\ R_{g1} & \cdots & R_{gk} \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ \vdots \\ r_g \end{bmatrix} \quad rank(R) = g$

Maximum Likelihood Estimator over entire parameter space: $\hat{\theta}$

Maximum Likelihood Estimator over constraint under H_0 : $\tilde{\theta}$

Likelihood Ratio Statistic: $X_{LR}^2 = -2\left[\ln\left(L\left(\tilde{\theta}, y\right)\right) - \ln\left(L\left(\hat{\theta}, y\right)\right)\right]$

Wald statistic: $X_W^2 = n_{\bullet}\left(R\hat{\theta} - r\right)^T \left(RI^{-1}\left(\hat{\theta}\right)R^T\right)^{-1}\left(R\hat{\theta} - r\right)$

Lagrange Multiplier (Score) Statistic: $X_{LM}^2 = \frac{1}{n_{\bullet}} s\left(\tilde{\theta}, y\right)^T \left(I\left(\tilde{\theta}\right)\right)^{-1} s\left(\tilde{\theta}, y\right)$

where: $I_{ij}(\theta) = -\frac{1}{n_{\bullet}} E\left[\frac{\partial^2}{\partial\theta_i \partial\theta_j} \ln\left(L(\theta, y)\right)\right] \qquad s_i(\theta, y) = \frac{\partial}{\partial\theta_i} \ln\left(L(\theta, y)\right)$

Soccer Goals Example

- Premier League Games in 2004 for k=5 European Countries:
  - England $n_1 = 380,\ Y_{1\bullet} = 975$
  - France $n_2 = 380,\ Y_{2\bullet} = 826$
  - Germany $n_3 = 306,\ Y_{3\bullet} = 890$
  - Italy $n_4 = 380,\ Y_{4\bullet} = 960$
  - Spain $n_5 = 380,\ Y_{5\bullet} = 980$

$$L(\theta, y) = \frac{\exp\left(-\sum_{i=1}^{5} n_i\theta_i\right)\prod_{i=1}^{5}\theta_i^{y_{i\bullet}}}{\prod_{i=1}^{5}\prod_{j=1}^{n_i} y_{ij}!} \qquad y_{i\bullet} = \sum_{j=1}^{n_i} y_{ij}$$

Testing Equality of Mean Goals Among Countries - I

$$H_0 : \theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 \quad \Rightarrow \quad R\theta = r \qquad R = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} \quad r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\ln\left(L(\theta, y)\right) = -\sum_{i=1}^{5} n_i \theta_i + \sum_{i=1}^{5} (y_{i\bullet}) \ln(\theta_i) - \ln\left(\prod_{i=1}^{5} \prod_{j=1}^{n_i} y_{ij}! \right)$$

$$\text{Under H}_0 : \ln\left(L(\theta, y)\right) = -\theta n_\bullet + (y_{\bullet\bullet}) \ln(\theta) - \ln\left(\prod_{i=1}^{5} \prod_{j=1}^{n_i} y_{ij}! \right)$$

$$\frac{\partial \ln\left(L(\theta, y)\right)}{\partial \theta_i} = -n_i + \frac{y_{i\bullet}}{\theta_i} \quad \Rightarrow \quad \hat{\theta}_i = \frac{y_{i\bullet}}{n_i} = \bar{y}_{i\bullet}$$

$$\text{Under H}_0 : \frac{\partial \ln\left(L(\theta, y)\right)}{\partial \theta} = -n_\bullet + \frac{y_{\bullet\bullet}}{\theta} \quad \Rightarrow \quad \tilde{\theta} = \frac{y_{\bullet\bullet}}{n_\bullet} = \bar{y}_{\bullet\bullet}$$

$$\Rightarrow \quad \hat{\theta}_1 = \frac{975}{380} = 2.57 \quad \hat{\theta}_2 = \frac{826}{380} = 2.17 \quad \hat{\theta}_3 = \frac{890}{306} = 2.91 \quad \hat{\theta}_4 = \frac{960}{380} = 2.53 \quad \hat{\theta}_5 = \frac{980}{380} = 2.58$$

$$\tilde{\theta} = \frac{975 + 826 + 890 + 960 + 980}{380 + 380 + 306 + 380 + 380} = \frac{4631}{1826} = 2.54$$

$$\frac{\partial^2 \ln\left(L(\theta, y)\right)}{\partial \theta_i^2} = -\frac{y_{i\bullet}}{\theta_i^2} \qquad \frac{\partial^2 \ln\left(L(\theta, y)\right)}{\partial \theta_i \partial \theta_j} = 0 \qquad E\left[\frac{\partial^2 \ln\left(L(\theta, y)\right)}{\partial \theta_i^2} \right] = -\frac{n_i \theta_i}{\theta_i^2} = -\frac{n_i}{\theta_i}$$

Testing Equality of Mean Goals Among Countries - II

$$s(\theta, y) = \begin{bmatrix} -n_1 + \dfrac{y_{1\bullet}}{\theta_1} \\ -n_2 + \dfrac{y_{2\bullet}}{\theta_2} \\ -n_3 + \dfrac{y_{3\bullet}}{\theta_3} \\ -n_4 + \dfrac{y_{4\bullet}}{\theta_4} \\ -n_5 + \dfrac{y_{5\bullet}}{\theta_5} \end{bmatrix} \qquad I(\theta, y) = \begin{bmatrix} \dfrac{380}{1826\theta_1} & 0 & 0 & 0 & 0 \\ 0 & \dfrac{380}{1826\theta_2} & 0 & 0 & 0 \\ 0 & 0 & \dfrac{306}{1826\theta_3} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{380}{1826\theta_4} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{380}{1826\theta_5} \end{bmatrix}$$

Likelihood Ratio Test

$$\ln\left(L\left(\hat{\theta}, y\right)\right) = -\sum_{i=1}^{5} n_i \hat{\theta}_i + \sum_{i=1}^{5} (y_{i\bullet}) \ln\left(\hat{\theta}_i\right) - \ln\left(\prod_{i=1}^{5}\prod_{j=1}^{n_i} y_{ij}!\right)$$

$$= -y_{\bullet\bullet} + \sum_{i=1}^{5} (y_{i\bullet}) \ln\left(\overline{y}_{i\bullet}\right) - \ln\left(\prod_{i=1}^{5}\prod_{j=1}^{n_i} y_{ij}!\right) =$$

$$= -4631 + (918.71 + 641.33 + 950.20 + 889.69 + 928.43) - \ln\left(\prod_{i=1}^{5}\prod_{j=1}^{n_i} y_{ij}!\right)$$

$$= -4631 + 4328.36 - \ln\left(\prod_{i=1}^{5}\prod_{j=1}^{n_i} y_{ij}!\right) = -302.64 - \ln\left(\prod_{i=1}^{5}\prod_{j=1}^{n_i} y_{ij}!\right)$$

$$\ln\left(L\left(\tilde{\theta}, y\right)\right) = -y_{\bullet\bullet} + \ln\left(\tilde{\theta}\right) y_{\bullet\bullet} - \ln\left(\prod_{i=1}^{5}\prod_{j=1}^{n_i} y_{ij}!\right) =$$

$$= -4631 + 4309.82 - \ln\left(\prod_{i=1}^{5}\prod_{j=1}^{n_i} y_{ij}!\right) = -321.18 - \ln\left(\prod_{i=1}^{5}\prod_{j=1}^{n_i} y_{ij}!\right)$$

$$\Rightarrow \quad X_{LR}^2 = -2\left[-321.18 - (-302.64)\right] = 37.08 \qquad \chi^2_{4,.05} = 9.49$$

Evidence that the true population means differ (in particular: France lower, Germany higher than the others)

Wald Test

Wald statistic: $X_W^2 = n_{\bullet}\left(R\hat{\theta}-r\right)^T\left(RI^{-1}\left(\hat{\theta}\right)R^T\right)^{-1}\left(R\hat{\theta}-r\right)$

$$R\hat{\theta}-r=\begin{bmatrix}1 & -1 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0\\ 1 & 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}2.57\\ 2.17\\ 2.91\\ 2.53\\ 2.58\end{bmatrix}-\begin{bmatrix}0\\0\\0\\0\end{bmatrix}=\begin{bmatrix}2.57-2.17-0\\ 2.57-2.91-0\\ 2.57-2.53-0\\ 2.57-2.58-0\end{bmatrix}=\begin{bmatrix}0.40\\ -0.34\\ 0.04\\ -0.01\end{bmatrix}$$

$$RI^{-1}\left(\hat{\theta}\right)R^T=\begin{bmatrix}1 & -1 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0\\ 1 & 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1826(2.57)}{380} & 0 & 0 & 0 & 0\\ 0 & \frac{1826(2.17)}{380} & 0 & 0 & 0\\ 0 & 0 & \frac{1826(2.91)}{306} & 0 & 0\\ 0 & 0 & 0 & \frac{1826(2.53)}{380} & 0\\ 0 & 0 & 0 & 0 & \frac{1826(2.58)}{380}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}$$

$$=1826\begin{bmatrix}.0068 & -0057 & 0 & 0 & 0\\ .0068 & 0 & -.0095 & 0 & 0\\ .0068 & 0 & 0 & -.0067 & 0\\ .0068 & 0 & 0 & 0 & -.0068\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}=1826\begin{bmatrix}.0125 & .0068 & .0068 & .0068\\ .0068 & .0163 & .0068 & .0068\\ .0068 & .0068 & .0135 & .0068\\ .0068 & .0068 & .0068 & .0136\end{bmatrix}$$

$$\Rightarrow\quad\left(RI^{-1}\left(\hat{\theta}\right)R^T\right)^{-1}=\frac{1}{1826}\begin{bmatrix}132.72 & -25.34 & -36.23 & -35.49\\ -25.34 & 89.96 & -21.80 & -21.36\\ -36.23 & -21.80 & 119.25 & -30.53\\ -35.49 & -21.36 & -30.53 & 117.44\end{bmatrix}\quad\Rightarrow\quad X_W^2=38.33$$

Lagrange Multiplier (Score) Test

Lagrange Multiplier (Score) Statistic: $X^2_{LM} = \frac{1}{n_{\bullet}} s\left(\tilde{\theta}, y\right)^T \left(I\left(\tilde{\theta}\right)\right)^{-1} s\left(\tilde{\theta}, y\right)$

$$\tilde{\theta} = \bar{y}_{\bullet\bullet} = \frac{4631}{1826} = 2.5361$$

$$s\left(\tilde{\theta}, y\right) = \begin{bmatrix} -380 + \frac{975}{\tilde{\theta}} \\ -380 + \frac{826}{\tilde{\theta}} \\ -306 + \frac{890}{\tilde{\theta}} \\ -380 + \frac{960}{\tilde{\theta}} \\ -380 + \frac{980}{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} 4.42 \\ -54.31 \\ 44.93 \\ -1.47 \\ 6.41 \end{bmatrix} \qquad I\left(\tilde{\theta}, y\right) = \begin{bmatrix} \frac{380}{1826\tilde{\theta}} & 0 & 0 & 0 & 0 \\ 0 & \frac{380}{1826\tilde{\theta}} & 0 & 0 & 0 \\ 0 & 0 & \frac{306}{1826\tilde{\theta}} & 0 & 0 \\ 0 & 0 & 0 & \frac{380}{1826\tilde{\theta}} & 0 \\ 0 & 0 & 0 & 0 & \frac{380}{1826\tilde{\theta}} \end{bmatrix}$$

$$\Rightarrow s\left(\tilde{\theta}, y\right) = \begin{bmatrix} 4.42 \\ -54.31 \\ 44.93 \\ -1.47 \\ 6.41 \end{bmatrix} \quad \Rightarrow \quad I\left(\tilde{\theta}, y\right)^{-1} = \begin{bmatrix} 12.19 & 0 & 0 & 0 & 0 \\ 0 & 12.19 & 0 & 0 & 0 \\ 0 & 0 & 15.13 & 0 & 0 \\ 0 & 0 & 0 & 12.19 & 0 \\ 0 & 0 & 0 & 0 & 12.19 \end{bmatrix}$$

$$\Rightarrow \quad X^2_{LM} = 36.83$$

Testing Goodness of Fit to Poisson Distribution

- All estimation and testing has assumed that number of goals follow Poisson distributions
- To test whether that assumption is reasonable, we compare the observed distributions of goals with what we would expect under the Poisson model
- We can check whether the observed mean and variance are similar (under Poisson model they are equal)
- We can also obtain a chi-square statistic by summing over range of goals: $(\text{observed\#}-\text{expected\#})^2/\text{expected\#}$ which under hypothesis of model fits is approximately chi-square with (# in range)-1 degrees of freedom

Distributions of Goals

	Observed					Expected (Truncated at 7)					Chi-Square Statistic				
Goals	England	France	Germany	Italy	Spain	England	France	Germany	Italy	Spain	England	France	Germany	Italy	Spain
0	30	54	18	36	29	29.2062	43.2279	16.6947	30.3822	28.8244	0.0216	2.6843	0.1021	1.0388	0.0011
1	79	82	43	85	73	74.9370	93.9639	48.5563	76.7549	74.3367	0.2203	1.5233	0.6358	0.8857	0.0240
2	99	110	66	85	96	96.1363	102.1239	70.6130	96.9536	95.8553	0.0853	0.6074	0.3014	1.4738	0.0002
3	67	57	77	78	79	82.2218	73.9950	68.4592	81.6451	82.4019	2.8180	3.9034	1.0655	0.1627	0.1404
4	61	51	54	49	60	52.7410	40.2105	49.7783	51.5653	53.1275	1.2933	2.8951	0.3580	0.1276	0.8890
5	24	15	29	20	28	27.0645	17.4810	28.9560	26.0541	27.4026	0.3470	0.3521	0.0001	1.4068	0.0130
6	11	4	13	20	8	11.5736	6.3330	14.0364	10.9701	11.7783	0.0284	0.8595	0.0765	7.4328	1.2120
7	6	6	4	4	6	6.1196	2.6648	8.9060	5.6747	6.2732	1.3558	7.0529	0.9482	0.3095	0.0842
8	2	1	1	1	1	#N/A	#N/A	#N/A	#N/A	#N/A					
9	1	0	1	1	0	#N/A	#N/A	#N/A	#N/A	#N/A					
10	0	0	0	1	0	#N/A	#N/A	#N/A	#N/A	#N/A					
Total Games	380	380	306	380	380					Chi-square	6.1697	19.8780	3.4876	12.8377	2.3640
Total Goals	975	826	890	960	980					CritVal	14.0671	14.0671	14.0671	14.0671	14.0671
Average	2.5658	2.1737	2.9085	2.5263	2.5789					P-Value	0.5201	0.0058	0.8365	0.0762	0.9370

$$X_{obs}^2 = \sum_{i=0}^{7} \frac{(\text{obs}_i - \text{exp}_i)^2}{\text{exp}_i} \overset{approx}{\sim} \chi_7^2$$

All leagues, except France, appear to be well described by the Poisson distribution. Especially England, Germany, and Spain