

# Chapter 05 - Amortization and Sinking Funds

## Section 5.1 - Amortization

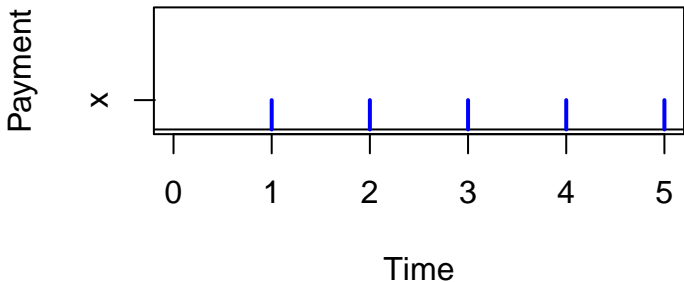
**Amortization Method** - The borrower repays the lender by means of installment payments at regularly spaced time points. The present value of the installment payments equals the

$$\text{Loan Principal} \quad L = (\text{Payment Amount}) \times a_{\overline{n}|i}$$

### Example:

\$1000 is borrowed with repayment by means of annual payments of  $x$  at the end of each of 5 years. The loan has an effective annual interest rate of 8%. What is the payment amount?

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Present value:  $1000 = xa_{\overline{5}|.08}$  produces

$$x = \frac{1000}{a_{\overline{5}|.08}} = \frac{1000(.08)}{1 - (1.08)^{-5}} = \$250.46.$$

as the amount of each payment.

## Section 5.2 - Outstanding Loan Balance

In the amortization method part of each payment pays interest on the loan and part of each payment repays some of the **principal of the loan** (the total amount borrowed). At a point in the repayment process we may need to ascertain the **outstanding loan balance** -

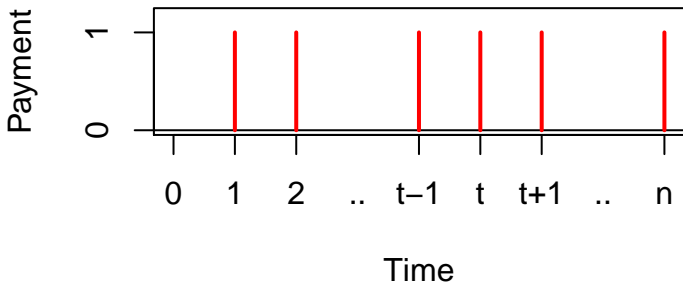
For example, if the loan needs to be refinanced or if the loan is to be purchased by another lender, it is vital to know how much of the original loan currently remains unpaid.

The outstanding loan balance can be determined in two ways:

**Prospectively** - The outstanding loan balance is **the present value of**

or

**Retrospectively** - The outstanding loan balance is **the original amount of the loan accumulated to the present date minus the accumulated value of all the loan payments that have already been paid.**



Suppose the payments are each 1 and the loan requires  $n$  payments. Let  $i$  denote the effective interest rate for each payment period (which is also the conversion period).

The loan amount is the present value at  $t = 0$ , namely

We seek the outstanding loan balance, denoted  $B_t$ , right after the  $t^{\text{th}}$  payment is made.

Prospective:

Retrospective:

$$\begin{aligned} B_t &= a_{\overline{n}|} (1+i)^t - s_{\overline{t}|} \\ &= \frac{(1-\nu^n)}{i} (1+i)^t - \frac{(1+i)^t - 1}{i} \\ &= \frac{1-\nu^{n-t}}{i} = a_{\overline{n-t}|} \end{aligned}$$

Thus either approach to this computation yields the same outstanding loan balance. If the loan is for  $L$  dollars, then the equal payment amounts should be

$$\frac{L}{a_{\overline{n}|}} \quad \text{dollars.}$$

Therefore, the outstanding loan balance right after the  $t^{\text{th}}$  payment is

**Example:**

A loan is created with 10 annual equal payments of \$500 at an effective annual rate of 6%. However, after 4 years, the borrower needs an additional \$2000 and must restructure all outstanding debts over the remaining 6 years at 7% effective. What is the payment amount during those 6 years?

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At 4 years the outstanding loan balance is  $500a_{\overline{6}|.06}$

The refinanced loan with payments of  $x$  dollars will have 6 payments and a present value at its beginning of

$$500a_{\overline{6}|.06} + 2000 = xa_{\overline{6}|.07} \quad \text{Therefore}$$

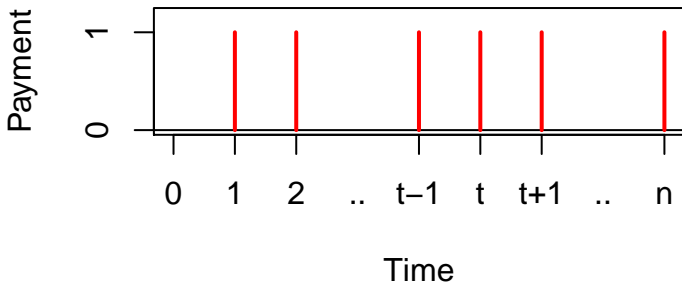
$$x = \frac{500a_{\overline{6}|.06} + 2000}{a_{\overline{6}|.07}} = \$935.41.$$

## Exercise 5-4:

A \$20,000 loan is to be repaid with annual payments at the end of each year for 12 years. If  $(1 + i)^4 = 2$ , find the outstanding loan balance immediately after the fourth payment.

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## Section 5.3 - Amortization Schedules



In the same setting as in the previous section,  $n$  total payments of 1 repay a loan of  $a_{\overline{n}|}$ . We now examine in greater detail the  $t^{\text{th}}$  payment.



Just after the  $(t - 1)^{th}$  payment, the outstanding loan balance is

$$B_{t-1} = a_{\overline{n-t+1}|}$$

So the interest due at the time of the  $t^{th}$  payment is

The remainder of this  $t^{th}$  payment of 1, namely  
is applied to the principal, reducing the outstanding loan balance to

$$\begin{aligned} a_{\overline{n-t+1}|} - v^{n-t+1} &= \frac{(1 - v^{n-t+1})}{i} - v^{n-t+1} \\ &= \frac{1 - (1 + i)v^{n-t+1}}{i} = \frac{1 - v^{n-t}}{i} \\ &= a_{\overline{n-t}|} \end{aligned}$$

which (as we saw in the previous section) is the outstanding loan balance right after the  $t^{th}$  payment.

## Generic Amortization Schedule

Payment index $t$	Payment amount 1	Interest paid $iB_{t-1}$	Principal repaid $1 - iB_{t-1}$	Outstanding loan balance $B_t$
0				$a_{\overline{n} }$
1	1	$1 - \nu^n$	$\nu^n$	$a_{\overline{n-1} }$
2	1	$1 - \nu^{n-1}$	$\nu^{n-1}$	$a_{\overline{n-2} }$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t^{th}$	1	$1 - \nu^{n-t+1}$	$\nu^{n-t+1}$	$a_{\overline{n-t} }$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n-1	1	$1 - \nu^2$	$\nu^2$	$a_{\overline{1} }$
n	1	$1 - \nu$	$\nu$	0
Total	$n$	$n - a_{\overline{n} }$	$a_{\overline{n} }$	

We see from this table that the total principal paid over all  $n$  payments is  $a_{\overline{n}|}$ , the amount of the original loan. Note also that the total interest paid is

In general for a loan of  $L$  dollars an amortization schedule is constructed by multiplying all the entries in the four main columns (not the index) by the payment amount  $\frac{L}{a_{\overline{n}|}}$ . Banks are willing to provide borrowers with the amortization schedule for their loan. It differs somewhat from our description due to roundoff issues, selection of a nice value for the payment amount and recursive generation of the entries in the table.

### Example:

Construct an amortization table for a loan of \$1000 to be paid in 4 annual payments at 10% annual effective interest rate.

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First note that the payment amount is:

$$L/a_{\overline{4}|.1} = 1000/3.169865 = 315.47.$$

The amortiation table is then

Payment index	Payment amount	Interest paid	Principal repaid	Outstanding loan balance
0				1,000.00
1	315.47	100.00	215.47	784.53
2	315.47	78.45	237.02	547.51
3	315.47	54.75	260.72	286.79
4	315.47	28.68	286.79	0.00
Total	1,261.88	261.88	1,000.00	

### Example:

Consider a home mortgage loan for \$100,000 at 6% nominal annual rate with equal monthly payments for 30 years. What are the characteristics of this loan?

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The effective monthly interest rate is  $i = .06/12 = .005$ .

The number of payments is  $n = 12(30) = 360$ .

Each monthly payment is  $\frac{100000}{a_{360|.005}} = \frac{100000}{166.7916} = 599.55$ .

At the end of the 30 years,

Total paid on the loan is  $\$599.55(360) = \$215,838$  with

total principal paid \$100,000 and

total interest paid \$115,838.

The amortization table for this loan is then

Payment index	Payment amount	Interest paid	Principal repaid	Outstanding loan balance
0				100,000.00
1	599.55	500.00	99.55	99,900.45
2	599.55	499.50	100.05	99,800.40
3	599.55	499.00	100.55	99,699.85
4	599.55	498.50	101.05	99,598.80
⋮	⋮	⋮	⋮	⋮
357	599.55	11.84	587.71	1,780.82
358	599.55	8.90	590.65	1,190.17
359	599.55	5.95	593.60	596.57
360	599.55	2.98	596.57	0.00
Total	215,838	115,838	100,000	

It is apparent in this example that to borrow \$100,000 the borrower must pay \$115,838. The **cost of the loan can be reduced** by, for example, cutting the term (time length) of the loan in half. This requires much larger monthly payment amounts.

An alternative method uses the amortization table, gradually increases the payment amounts and also cuts the loan term in half. With this method, the borrower makes only the odd numbered loan payments on the amortization schedule plus the principal on the next even numbered payment. Using the previous example, the payments change as shown below:

Payment index	Payment amount	Outstanding loan balance
0		100,000.00
1	$599.55 + 100.05 = 699.60$	99,800.40
2	$599.55 + 101.05 = 700.60$	99,598.80
⋮	⋮	⋮
179	$599.55 + 590.65 = 1,190.20$	1,190.17
180	$599.55 + 596.57 = 1,196.12$	0.00

This method is only valid in settings in which the loan has no prepayment penalty. But it results in saving the borrower \$57,794.36 in interest charges compared to the original 30 year loan.

## Exercise 5-14:

A 35 year loan is to be repaid with equal installments at the end of each year. The amount of interest paid in the 8<sup>th</sup> installment is \$135. The amount of interest paid in the 22<sup>nd</sup> installment is \$108. Calculate the amount of interest paid in the 29<sup>th</sup> installment.

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## Section 5.4 - Sinking Fund Method

In the sinking fund method of repaying a loan, the borrower agrees to make periodic interest payments on the complete loan amount and to repay the loan principal at the end of the loan term. In order to repay the principal at the end, the borrower is required to make periodic deposits in an interest earning account (called a **sinking fund**) sufficient to accrue the original loan amount at the end of the loan term.

Case (1) Suppose

$i$  = effective interest on the loan per payment period  
= effective interest earned in the sinking fund per pay period.

If  $L$  is the original loan amount  $i(L)$  is the interest due per payment period. Since periodic payments of 1 accumulate to  $s_{\overline{n}|i}$  over the  $n$  interest periods of the loan, then periodic payments of  $\frac{L}{s_{\overline{n}|i}}$  will accumulate to  $L$  at the end of the loan.

The required periodic payment is therefore

(See page 3-8.) We note that in this case,  $L/a_{\overline{n}|i}$  is also the constant periodic payment required under the amortization method, i.e. the two methods of structuring the loan require the same periodic payment amount.

**Example:** Using the setting of the first example of section 5.3, i.e.  $L = 1000$  and  $i = .10$ , the sinking fund structure produces:

Payment Index	Payment Amount	Interest Paid	Deposit in Sinking Fund	Sinking Fund Balance
0				0.00
1	315.47	100.00	215.47	215.47
2	315.47	100.00	215.47	452.49
3	315.47	100.00	215.47	713.21
4	315.47	100.00	215.47	1,000.00
Total	1,261.88	400.00	861.88	

## Case (2) Suppose

$i$  = effective interest on the loan per payment period

$j$  = effective interest earned in the sinking fund per pay period.

Typically  $i > j$ . If  $L$  is the original loan amount,

$iL$  = amount of interest due per payment period

and periodic deposits of  $\frac{L}{s_{\bar{n}|j}}$  will accumulate to  $L$  after  $n$  periods.

Putting these two together means that the borrower must set aside total payment amounts of

per period to cover both needs. Note that

$$i + \frac{1}{s_{\bar{n}|j}} = (i - j) + \frac{1}{a_{\bar{n}|j}}.$$

An amortization method approach with the same total payment amount would use an interest rate of  $i'$  that satisfies:

$$\frac{1}{a_{\bar{n}|i'}} = (i - j) + \frac{1}{a_{\bar{n}|j}} \quad \text{or}$$

$$a_{\bar{n}|i'} = \frac{a_{\bar{n}|j}}{(i - j)a_{\bar{n}|j} + 1}.$$

Example:

In the same setting as the previous example, consider a loan of \$1000 to be paid in 4 annual payments at 10% annual effective interest rate on the loan but with a growth rate of 7% annual effective interest rate in the sinking fund. The total payment required is

$$1000 \left[ (.10 - .07) + \frac{1}{a_{\bar{4}|.07}} \right] = 325.23.$$

The payment schedule would then be:

Payment Index	Payment Amount	Interest Paid	Deposit in Sinking Fund	Sinking Fund Balance
0				0.00
1	325.23	100.00	225.23	225.23
2	325.23	100.00	225.23	466.23
3	325.23	100.00	225.23	724.10
4	325.23	100.00	225.23	1,000.02
Total	1,300.92	400.00	900.92	

If this same payment \$325.23 was used in the amortization method, it would correspond to

$$a_{4|i'} = \frac{a_{4|.07}}{(.10 - .07)a_{4|.07} + 1} = 3.07476 \quad \text{or}$$

$$i' = .11421.$$

## Exercise 5-20:

A borrower is repaying a loan with 10 annual payments of \$1000. Half of the loan is repaid by the amortization method at 5% effective. The other half is repaid by the sinking fund method in which the lender receives 5% effective on the investment and the sinking fund accumulates at 4% effective. Find the amount of the loan to the nearest dollar.

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## Exercise 5-27:

A borrower is repaying a loan with payments of \$3000 at the end of every year over an unknown period of time. If the amount of interest in the third installment is \$2000, find the amount of principal in the sixth installment. Assume that interest is 10% convertible quarterly.

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## Section 5.6 - Varying Payments

Let  $R_t$  denote the  $t^{\text{th}}$  loan payment amount made at the end of period  $t$  and  $i$  denote the effective interest rate on the loan per payment period. The amount of the loan  $L$  is then

$$L = \sum_{t=1}^n v^t R_t$$

The outstanding loan balance just after the  $t^{\text{th}}$  payment is:

$$B_t =$$

The  $t^{\text{th}}$  payment of  $R_t$  is divided between

Interest:

and Principal:  $P_t = R_t - I_t$ .



Now consider the sequence of payments  $R_t$  when using the **sinking fund method** with  $i$  the effective rate of interest on the loan per payment period and  $j$  the effective rate of growth in the sinking fund per payment period. Each payment is divided between

$$\text{Interest: } I_t = iL$$

$$\text{and Principal: } P_t = R_t - iL.$$

Since the sinking fund must have an accumulated value of  $L$  at time  $t = n$ ,

$$\begin{aligned} L &= (R_1 - iL)(1 + j)^{n-1} + (R_2 - iL)(1 + j)^{n-2} + \cdots + (R_n - iL) \\ &= \sum_{t=1}^n (R_t - iL)(1 + j)^{n-t} \\ &= \sum_{t=1}^n R_t(1 + j)^{n-t} - iL \sum_{t=1}^n (1 + j)^{n-t} \quad \text{or} \end{aligned}$$

$$L = \sum_{t=1}^n R_t(1+j)^{n-t} - iLs_{\overline{n}|j}$$

which is the accumulated value of the loan payments minus the accumulated value of the interest payments. Solving this equation for  $L$  produces

$$\begin{aligned} L &= \frac{\sum_{t=1}^n R_t(1+j)^{n-t}}{1 + is_{\overline{n}|j}} \\ &= \frac{\sum_{t=1}^n R_t(1+j)^{n-t} \left( \frac{a_{\overline{n}|j}}{s_{\overline{n}|j}} \right)}{1 + (i-j)a_{\overline{n}|j}} \end{aligned}$$

### Example:

Using the amortization method a person borrows \$5000 at an effective rate of 8% per annum and agrees to repay the loan with payments at the end of each year. The first payment is \$600 and each subsequent payment is 4% above the previous one, with a smaller payment at the end of the term.

(a) What is the outstanding loan balance at the end of 5 years?

$$\begin{aligned} B_5 &= \\ &= (1 + .08)^5(5000) - \frac{(1.08)^5}{(1.04)} 600 \sum_{t=1}^5 \left(\frac{1.04}{1.08}\right)^t \\ &= 7346.64 - (847.689275) \frac{\left(\frac{1.04}{1.08}\right) \left[1 - \left(\frac{1.04}{1.08}\right)^5\right]}{1 - \left(\frac{1.04}{1.08}\right)} \\ &= 7346.64 - 3790.13 = \$3,556.51. \end{aligned}$$

(b) What is the principal and interest paid in the 5<sup>th</sup> payment?

5<sup>th</sup> payment amount:

$$600(1 + .04)^4 = 701.92.$$

Also the outstanding loan balance at  $t = 4$  is:

$$\begin{aligned} B_4 &= B_5(1.08)^{-1} + (701.92)(1.08)^{-1} \\ &= 3,942.99. \end{aligned}$$

Thus the interest and principal in the 5<sup>th</sup> payment are:

$$I_5 = (.08)(3942.99) = \$315.44 \quad \text{and}$$

$$P_5 = 701.92 - 315.44 = \$386.48 \quad \text{respectively.}$$

### Exercise 5-33:

A 10-year loan of \$2000 is to be repaid with payments at the end of each year. It can be repaid under two options:

- (a) Equal annual payments at an annual effective rate of 8.07%, or
- (b) Installments of \$200 each year plus interest on the unpaid balance at an annual effective rate of  $i$ .

The sum of the payments under both options is the same.

Determine  $i$ .

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## Exercise:

Judy buys an \$8000 car on a 4 year "lease with option to buy" arrangement which requires her to pay \$150 per month to cover the interest (12% convertible monthly) plus a portion of the principal. Judy sets up a sinking fund ( 9% convertible monthly ) to accumulate the remaining principal at the lease end. What should she deposit in the sinking fund at the end of each month?

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