Example 1 Find the factor level means and main effects for the following set of **true** cell means:

	Fertilizer	
Manure	Low	High
Low	11.0	14.0
High	14.0	15.0

Example 1. Table of cell means

	Fertilizer	
Manure	Low	High
Low	11.0	14.0
High	14.0	15.0

Important concept: There is *interaction* in this model because $\mu_{21} - \mu_{11} = 14 - 11 = 3$, while $\mu_{22} - \mu_{12} = 15 - 14 = 1$.

That is, the effect of HIgh vs. Low Manure is not the same at both Low and High levels of Fertilizer. We say that the two factors, Manure and Fertilizer, interact.

Exercise Are the effects of HIgh vs. Low Fertilizer the same at Low and High Manure? Use the model notation to explain your answer.

Example 2 Suppose the true cell means were the following:

	Fertilizer		Row	Main row
Manure	Low	High	Average	effect
Low	12.0	14.0	13.0	-2.0
High	16.0	18.0	17.0	2.0
Column average	14.0	16.0	15.0	
Main col. effect	-1.0	1.0		

In this special case, the factor effects are **additive.** This means, for instance, that each cell mean can be found by adding the respective row and column effects to the overall mean $\mu_{..}$.

$$\mu_{11} = \mu_{..} + \alpha_1 + \beta_1 = 15 + (-2) + (-1)$$
$$\mu_{12} = \mu_{..} + \alpha_1 + \beta_2 = 15 + (-2) + 1$$
$$\vdots$$

This also means that $\mu_{ij} = \mu_{i.} + \mu_{.j} - \mu_{..}$.

This is an example of an **additive** model. We also say there is **no interaction** between the two factors.

Example 2

Manure	Low	High
Low	12.0	14.0
High	16.0	18.0

Compare *simple effects* of Manure at all levels of Fertilizer. Is $\mu_{21} - \mu_{11} = \mu_{22} - \mu_{12}$? Answer is "Yes, $4 = 4$." So, by our first definition of interaction, there is no interaction between Manure and Fertilizer in this case. This is an *additive model*.

Another definition of additivity is based on using the main effects and grand mean. Does $\mu_{ij} = \mu_{..} + \alpha_i + \beta_j$ for every pair (i, j)? If yes, then the model is additive. Here, again, the answer is "Yes."

The *interaction plot* is a useful plot of the cell means which enables us to easily spot whether there is any interaction, and to diagnose the nature of any interaction that exists.

Example 2

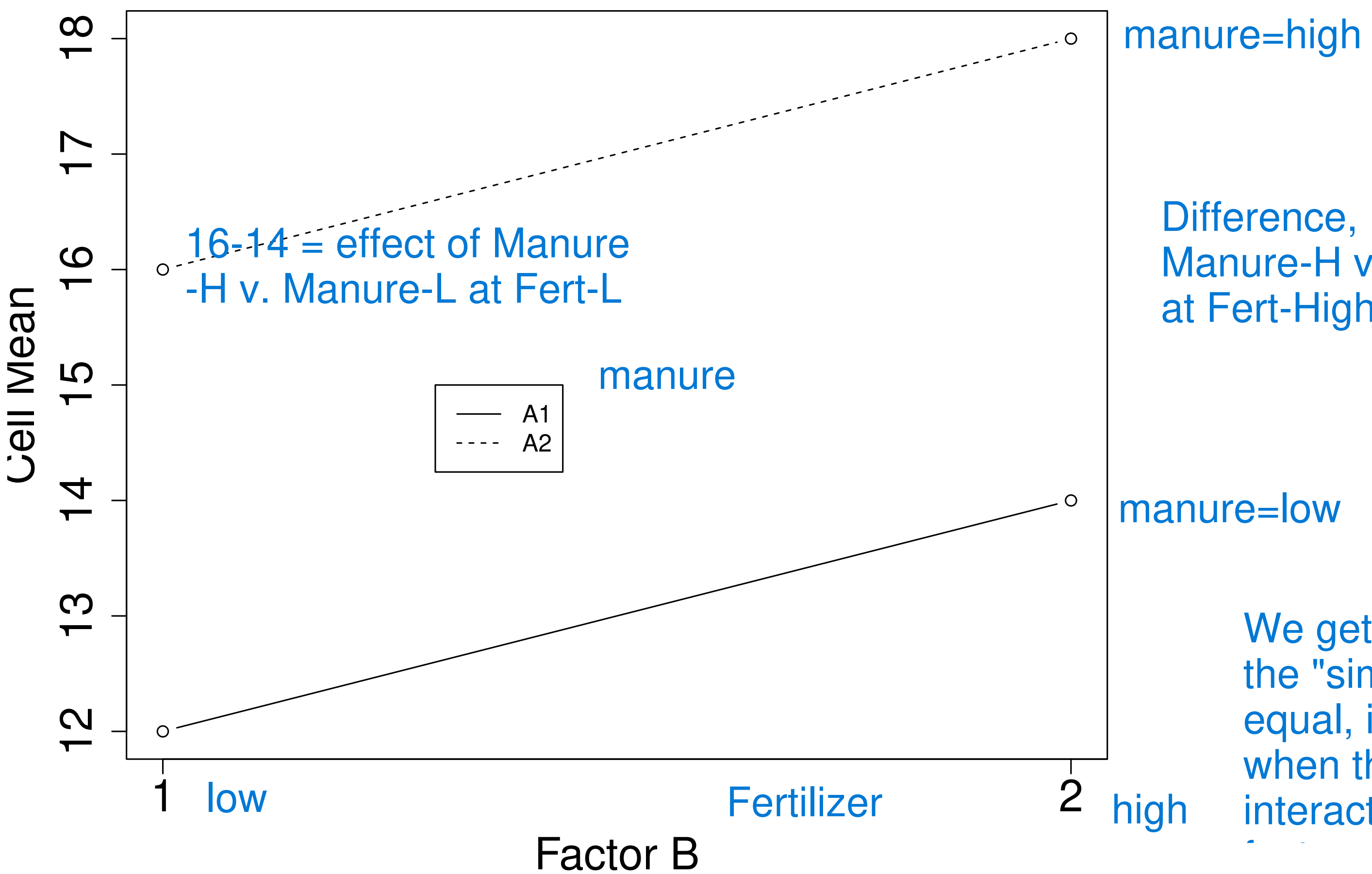
	Fertilizer	
Manure	Low	High
Low	12.0	14.0
High	16.0	18.0

Interaction plot (true cell means)

Interaction Plot is of **treatment** means with levels of one factor represented on the horizontal axis, and treatment means for the levels of the other factor connected by line segments.

Interaction plot, additive model (Example 2):

Note: Consider another example where the lines actually cross, by tweaking these numbers. (Done in class) Worst type of interaction.



Difference, 18-14, = effect of Manure-H v. Manure-L at Fert-High

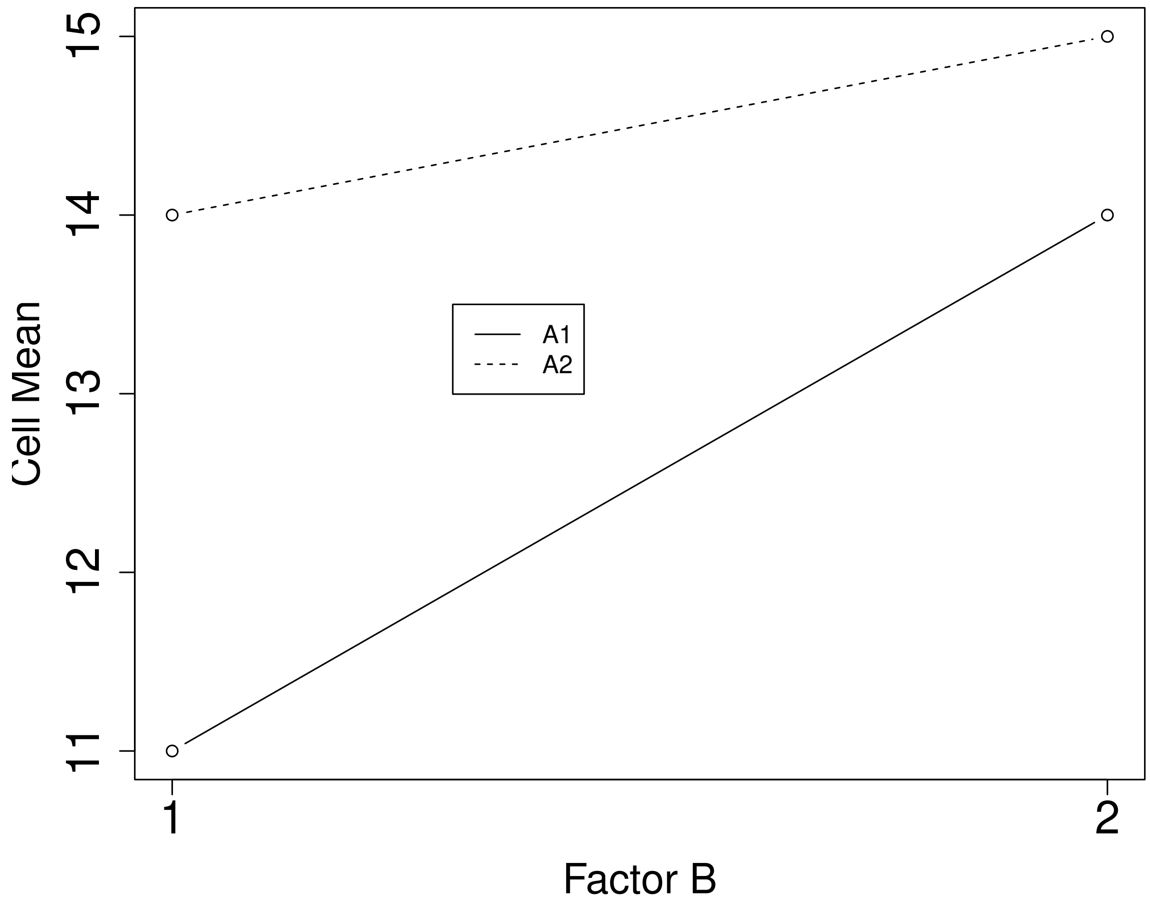
We get parallel lines when the "simple effects" are all equal, in other words, when there is no interaction between the

Interaction Plot, Example 2

The lines are parallel. This means the effect of Factor A is constant for all levels of Factor B.

So, Factors A and B do not interact.

Interaction plot, model with interaction (Example 1):



Nonparallel lines indicate the two factors interact. The simple effects of Factor A (Manure) are not the same for the two levels of Factor B (Fertilizer)

Interaction plot, model with interaction (Example 1):
The lines are not parallel. So, we do have interaction between Factors A and B in their effects on the response Y.

Let $\alpha_i = \mu_{i.} - \mu_{..}$ represent the i^{th} row effect, and $\beta_j = \mu_{.j} - \mu_{..}$ represent the j^{th} column effect:

If for all i, j, the cell mean $\mu_{ij} = \mu_{..} + \alpha_i + \beta_j$, then the factor effects are additive. Otherwise, the factor effects are interacting.

Interaction of i^{th} level of Factor A with j^{th} level of Factor B is defined to be

$$(\alpha\beta)_{ij} = \mu_{ij} - (\mu_{..} + \alpha_i + \beta_j)$$

Example (p. 111)

	Fertilizer		Row	Main row
Manure	Low	High	Average	effect
Low	11.0	14.0	12.5	-1.0
High	14.0	15.0	14.5	1.0
Column avg.	12.5	14.5	$\mu_{..} = 13.5$	
Main col. effects	-1.0	1.0		

$$(\alpha\beta)_{11} = \quad 11.0 - (13.5 + (-1.0) + (-1.0)) = 11.0 - 11.5 = -.5$$

$$(\alpha\beta)_{12} = \quad 14.0 - (13.5 + (-1.0) + (1.0)) = 14.0 - 13.5 = .5$$

$$\vdots$$

There are several ways to recognize whether or not interaction is present. All of the following four ways are equivalent:

1. By examining whether all μ_{ij} can be expressed as the sums $\mu_{..} + \alpha_i + \beta_j$.
2. By examining whether the difference between the mean responses for any two levels of factor B is the same for all levels of factor A.
3. By examining whether the difference between the mean responses for any two levels of factor A is the same for all levels of factor B.
4. By examining whether the treatment means curves for the different factor levels in an interaction plot are parallel.

If any one of the above four conditions holds, then they all hold, and there is no interaction.

2. Ex. 2, chart shows effect of A (manure) (A2 - A1) is 2 for each level of B (fertilizer)

3. Ex. 2, chart shows effect of B (fertilizer, H - L) is 2 for each level of A (Manure).

4. Difference between the two lines is the same for each level of B, so effect of A (A2 - A1) is the same for each level of B.

Could draw the interaction plot with lines connecting the means for levels of A.

Important vs. Unimportant Interactions

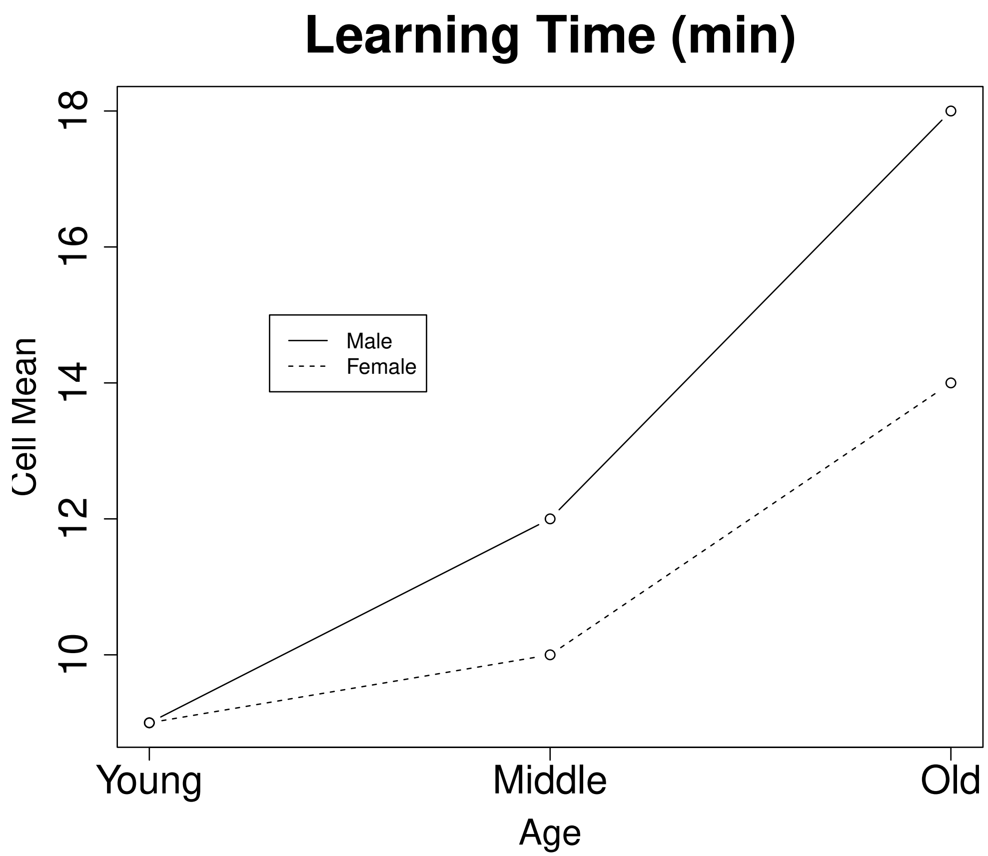
When interaction is present, it is generally the case that factor level means are not useful. Rather, the effects of Factor B need to be reported separately for each level of Factor A (and vice versa, the effects of Factor A are reported separately for each level of Factor B).

Sometimes, the observed interaction is so small that it is still meaningful to report main effects. We say the interaction is “unimportant” in this case.

Example Learning time for Males and Females in Three Age Groups (Section 19.2) completely observational study; both factors are observational

Consider a two-factor observational study, in which the effects of gender (male, female) and age (young, middle, old) on learning time (measured in minutes) are of interest.

Example Interaction in Learning Time study, Case 1



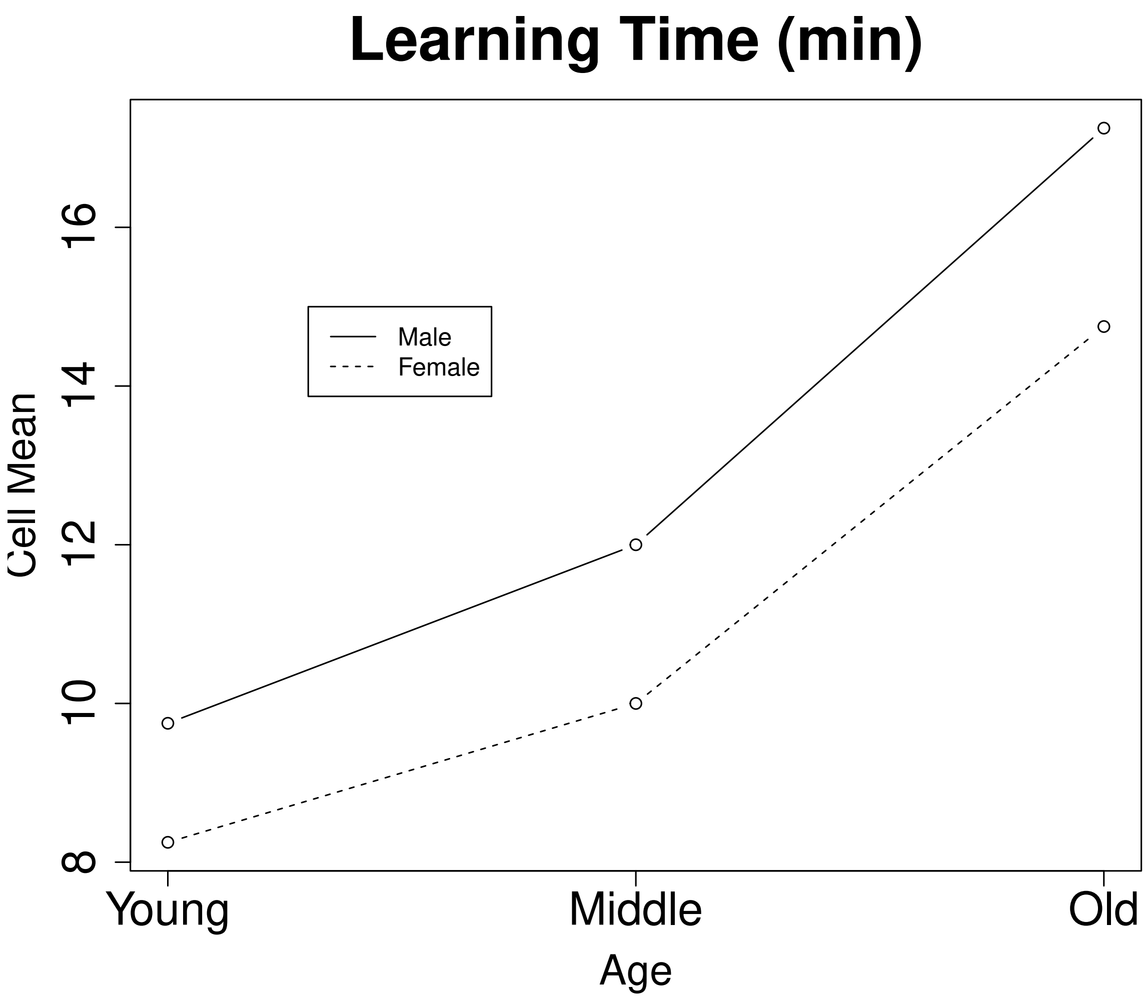
Discuss: Is interaction important here? Why or why not?

Yes, interaction is important because there is no difference between Young Males and Females in mean learning time, but Males take longer than Females on average at both the higher ages.

Because the interaction is important, in the analysis you would want to report effect of Gender separately for the three age groups.

Discuss: What if anything would you miss, if you reported the average for Males, over the three age groups, minus the average for Females, over the three age groups?

Example Interaction in Learning Time study, Case 2



Not important interaction, because the increased difference in learning times between Males and Females as Age increases, is slight.