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Author(s): Alan Agresti
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Measures of Nominal-Ordinal Association

ALAN AGRESTI*

Measures are formulated for summarizing the strength of association between a nominal variable and an ordered categorical variable. The measures are differences or ratios of probabilities of events concerning two types of pairs of observations. They can be used to describe the degree of difference between two or more groups on an ordinal response variable. The measures summarize and complement the results of fitting models to nominal-ordinal cross-classification tables, especially when a single structural model form cannot be found that adequately describes an entire table or set of tables.

KEY WORDS: Cross-classification tables; Somers's d, Gamma; Freeman's theta; Mann-Whitney statistic; Logit and loglinear models.

1. INTRODUCTION

The most appropriate measures for summarizing the degree of association between two variables depend on the measurement scales of those variables. In this article we propose some summary measures of the degree of association between a nominal variable and an ordinal variable. These might be used to describe associations between pairs of variables such as religious affiliation and opinion about abortion, marital status and life satisfaction, race and severity of criminal punishment, and type of medical treatment and degree of recovery from disease. In most applications in which this combination of measurement scales occurs, the ordinal variable is naturally regarded as a response variable. One such setting is when the levels of the nominal variable represent r groups (e.g., religious types, races, regions) that we want to compare with respect to their distribution on an ordered categorical response. To reflect this common direction in the relationship, the measures we propose are asymmetric in nature. We only consider discrete ordinal variables, for which case the data may be summarized in a cross-classification table having r unordered rows and c ordered columns.

In analyzing ordinal variables such as the ones just mentioned, different researchers would probably use quite different scoring patterns if asked to assign numerical values to the levels. Also, it is often advantageous to be able to present summaries or to make conclusions that are not founded on scoring systems or strong distributional assumptions concerning the variables. Because of these considerations, the measures of nominal-ordinal association we propose use only the ordering of the levels of the ordinal variable.

We start by considering the $2 \times c$ table in Section 2. For this case, the measures we suggest are related to more familiar measures of association, nonparametric test statistics, and ridit measures proposed in other contexts. In Section 3 we construct two generalized measures for the $r \times c$ case that can be expressed in terms of probabilities of events concerning two types of pairs of members. One of these measures is an alternative representation of Freeman's (1965) theta index.

Several authors have proposed various types of models for describing cross-classifications of nominal and ordinal variables. A well-fitting model can be used to test the null hypothesis of independence against meaningful alternatives, and its structural form describes the nature of the bivariate relationship. Our emphasis in this article is on developing summary measures of the strength of the association in order to complement such tests and many such models.

Table 1 contains a cross-classification of the estimated United States population in 1975 by region and by size of residential area. In Section 4.3 we illustrate how the measures of association presented in this article can provide useful summaries for these data. The complementarity of these measures to the model-building process is especially important for data like these, which we will see are poorly fit by unsaturated loglinear and logit models for this setting.

2. DICHOTOMOUS NOMINAL VARIABLE

Suppose that sampling units may be classified on a dichotomous nominal variable and on an ordinal variable having c categories labeled $1, 2, \ldots, c$ from least to greatest in degree. Measures of nominal-ordinal association then correspond to measures of the difference between two groups in the distribution of an ordinal variable. Let ρ_{ij} denote the probability that a randomly selected individual is classified in level i of the nominal variable (which we refer to as group i, $i = 1, 2$) and level j of the ordinal variable ($j = 1, 2, \ldots, c$). We let $\tilde{\rho}_{ij} = \rho_{ij}/\rho_{i\cdot}$ be the conditional probability that a member is classified in level j of the ordinal variable, given membership in the ith group. Finally, let Y_1 and Y_2 be independent random variables giving the category numbers

* Alan Agresti is Associate Professor, Department of Statistics, University of Florida. The author is grateful to the referees and the editor for helpful comments. Part of this research was completed while the author was visiting the Department of Statistics at Oregon State University.

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Table 1. Population Distribution (in thousands) in 1975 by Region and Size of Residential Area, With Statistics for Loglinear and Logit Models

Region	Size of Residential Area		
	Non-metropolitan	Other Metropolitan	Large Metropolitan
North	17,763	17,290	22,612
South	24,555	28,546	15,000
	Expected Frequencies for Loglinear Model and Log Odds of Adjacent Cell Frequencies		
North	15,842.1(.03)	21,131.7(−.27)	20,691.1
South	26,475.9(−.15)	24,704.3(.64)	16,920.8
	Expected Frequencies for Logit Model and Observed Logits		
North	15,815.7(−.81)	21,184.6(.44)	20,664.7
South	26,502.3(−.57)	24,651.4(1.26)	16,947.3

NOTE: For simplicity of illustration, we list data for only two of the four regions given in the original table.

Source: U.S. Bureau of the Census (1977), *Current Population Reports*, p-25, no. 709, Table A.

of the ordinal variable for members selected at random from group 1 and group 2, respectively.

2.1 Delta

A simple measure of association that uses only the ordering of the levels of the ordinal variable is

$$\delta = P(Y_1 > Y_2) - P(Y_2 > Y_1) = \sum_{i>j} \tilde{\rho}_{1i}\tilde{\rho}_{2j} - \sum_{i<j} \tilde{\rho}_{1i}\tilde{\rho}_{2j}. \tag{2.1}$$

Clearly, $-1 \leq \delta \leq 1$, with $|\delta| = 1$ if and only if one of the $\{\tilde{\rho}_{ij}, 1 \leq j \leq c\}$ distributions is entirely below or entirely above the other. When there are but $c = 2$ response categories, $\delta = \tilde{\rho}_{21} - \tilde{\rho}_{11}$, the standard difference of proportions.

Our interest in $P(Y_1 > Y_2) - P(Y_2 > Y_1)$ instead of (say) $P(Y_1 > Y_2)$ alone is so that the range of possible values is centered around a number (zero) that always occurs when the two variables are independent. Alternatively, one might use the measure

$$P(Y_1 > Y_2) + P(Y_1 = Y_2)/2 = (\delta + 1)/2, \tag{2.2}$$

which takes on values between 0 and 1 and equals .5 when the variables are independent (see Klotz 1966).

Not surprisingly, δ is a special case of other descriptive measures commonly used for ordered categorical data. The asymmetric ordinal measure of association, Somers's d (Somers 1962), is defined to be the difference between the proportion of concordant pairs and the proportion of discordant pairs, out of those pairs of members that are untied on the independent variable. In this setting, if the dichotomous variable is treated as an independent variable, with group 1 arbitrarily considered to be the higher level, Somers's d equals δ. The ridit measure introduced by Bross (1958) is also related to δ. Suppose that ridit scores are assigned to the c categories of the ordinal variable by treating $\{\tilde{\rho}_{2j}\}$ as the "identified distribution." Then the mean ridit score for the $\{\tilde{\rho}_{1j}\}$ distribution is $(\delta + 1)/2$, precisely the measure defined in (2.2). The relationship between ridit analysis and Somers's d measure was noted by Vigderhous (1979).

Given random samples of sizes n_1 and n_2 from the two groups and frequencies $\{n_{ij}\}$ in the cells, a sample analog of δ is

$$\hat{\delta} = \Big(\sum_{i>j} n_{1i}n_{2j} - \sum_{i<j} n_{1i}n_{2j}\Big)/n_1n_2 = (U - U')/n_1n_2, \tag{2.3}$$

where $U = \sum_{i>j} n_{1i}n_{2j}$ and $U' = \sum_{i<j} n_{1i}n_{2j}$ are discrete analogs of the statistics on which the Mann-Whitney test is based for continuous data. Equivalently, $\hat{\delta} = [S_1 - E(S_1)]/(n_1n_2/2)$, where S_1 denotes the rank sum for the first group (average ranks being assigned to the levels of the ordinal variable) and $E(S_1) = n_1(n_1 + n_2 + 1)/2$ denotes the expected value of S_1 when $\{\tilde{\rho}_{1j} = \tilde{\rho}_{2j}, j = 1, \ldots, c\}$. Thus, $\hat{\delta}$ may be interpreted as the difference between S_1 and its expected value when the distributions are identical, divided by the maximum possible value of that difference. It follows that $\hat{\delta} > 0$ if and only if the mean rank for group 1 exceeds the mean rank for group 2.

2.2 Alpha

In some applications, it is informative to describe the relative sizes of $P(Y_1 > Y_2)$ and $P(Y_2 > Y_1)$ in ratio form,

$$\alpha = P(Y_1 > Y_2)/P(Y_2 > Y_1) = \sum_{i>j} \tilde{\rho}_{1i}\tilde{\rho}_{2j} \Big/ \sum_{i<j} \tilde{\rho}_{1i}\tilde{\rho}_{2j} = \sum_{i>j} \rho_{1i}\rho_{2j} \Big/ \sum_{i<j} \rho_{1i}\rho_{2j}. \tag{2.4}$$

We see that $0 \leq \alpha \leq \infty$, with $\alpha - 1$ having the same sign as δ. Note that $\alpha = 0$ if $\delta = -1$ and $\alpha = \infty$ if $\delta = 1$, but the reverse implications do not hold unless $P(Y_1 = Y_2) = 0$.

The sample version $\hat{\alpha}$ of α is related to the Mann-Whitney statistics by $\hat{\alpha} = U/U'$. For the special case in which there are only $c = 2$ response categories, α reduces to the odds ratio, $\rho_{12}\rho_{21}/\rho_{11}\rho_{22}$. Like the odds ratio, it is often useful to measure α on the logarithmic scale, since $\ln(\alpha)$ is symmetric around the independence value of zero, and since the distribution of its sample analog converges to normality faster than the distribution of $\hat{\alpha}$. When the two groups are themselves naturally ordered, Goodman and Kruskal's (1954) gamma measure equals $(\alpha - 1)/(\alpha + 1)$.

3. NOMINAL-ORDINAL ASSOCIATION

We now generalize α and δ in order to describe the degree of association between an ordered categorical response variable Y and a nominal variable X having r levels. For the population of interest, let ρ_{ij} denote the probability that a randomly selected member is classified in level i of X and level j of Y and let $\tilde{\rho}_{ij} = \rho_{ij}/\rho_{i\cdot}$, $1 \le i \le r$, $1 \le j \le c$.

3.1 Generalized Delta

We define a generalized version of δ as a difference of probabilities of two types of pairs, so that it shares the simple structure exhibited by δ in the dichotomous ($r = 2$) case and by ordinal measures such as Kendall's tau and gamma.

Let δ_{ik} denote the value of δ for the $2 \times c$ table obtained by considering levels i and k of the nominal variable X as groups 1 and 2, respectively. Let y_i be the category number of the ordinal variable Y for a member selected at random from the ith level of X, $i = 1, 2, \ldots, r$. The pair of responses y_i and y_k has *consistent order* if $y_i - y_k$ has the same sign as δ_{ik}. A pair has *inconsistent order* if it has the opposite sign. Let C and I denote consistent order and inconsistent order, respectively, of a randomly selected pair, and let the symbol U_X mean that the members are classified in different levels of X. Finally, let

$$G_i^+ = \{k: \delta_{ik} > 0\},$$

$$G_i^- = \{k: \delta_{ik} < 0\},$$

$$R_{ij}^{(C)} = \sum_{k \in G_i^+} \sum_{l<j} \rho_{kl} + \sum_{k \in G_i^-} \sum_{l>j} \rho_{kl}, \tag{3.1}$$

and

$$R_{ij}^{(I)} = \sum_{k \in G_i^+} \sum_{l>j} \rho_{kl} + \sum_{k \in G_i^-} \sum_{l<j} \rho_{kl}.$$

Notice that $R_{ij}^{(C)}$ is the probability that a pair of members, one in level i of X and level j of Y and the other randomly chosen, will have consistent order. It follows that

$$P(C) = \sum_{i,j} \rho_{ij} R_{ij}^{(C)},$$

$$P(I) = \sum_{i,j} \rho_{ij} R_{ij}^{(I)}, \tag{3.2}$$

and

$$P(U_X) = 2 \sum_{i<k} \rho_{i\cdot}\rho_{k\cdot}.$$

We now define generalized delta as

$$\bar{\delta} = P(C \mid U_X) - P(I \mid U_X) \tag{3.3}$$
$$= \sum_{i,j} \rho_{ij}(R_{ij}^{(C)} - R_{ij}^{(I)})/2 \sum_{i<k} \rho_{i\cdot}\rho_{k\cdot}.$$

When $r = 2$, note that $\bar{\delta} = |\delta|$ as defined in (2.1).

This measure can be shown to equal a weighted average of the absolute values of the δ values for the $\binom{r}{2}$ $2 \times c$ tables representing the various pairs of levels of X; namely,

$$\bar{\delta} = \sum_{i<k} \rho_{i\cdot}\rho_{k\cdot} \, |\delta_{ik}| / \sum_{i<k} \rho_{i\cdot}\rho_{k\cdot}. \tag{3.4}$$

The weight assigned to the $|\delta_{ik}|$ for a particular pair of levels of X is the probability that two levels of X chosen at random (according to the $\{\rho_{i\cdot}\}$ distribution) would yield that pair, given that they are different. It follows from this representation that $0 \le \bar{\delta} \le 1$, with $\bar{\delta} = 0$ iff all $\delta_{ik} = 0$ and $\bar{\delta} = 1$ iff all $|\delta_{ik}| = 1$. This measure can be easily generalized to describe partial association by forming a weighted average of the $\bar{\delta}$ values that are computed within combinations of levels of control variables.

Under full or independent multinomial sampling, the sample analog of $\bar{\delta}$ is asymptotically normally distributed. The asymptotic variance formulas for the measures presented in this paper are given in the Appendix.

3.2 Generalized Alpha

In generalizing from two to several levels of X, we extended $\delta = P(Y_1 > Y_2) - P(Y_2 > Y_1)$ to $\bar{\delta} = P(C \mid U_X) - P(I \mid U_X)$. Similarly, we generalize $\alpha = P(Y_1 > Y_2)/P(Y_1 < Y_2)$ to

$$\bar{\alpha} = P(C \mid U_X)/P(I \mid U_X)$$
$$= \sum_{i,j} \rho_{ij} R_{ij}^{(C)} / \sum_{i,j} \rho_{ij} R_{ij}^{(I)} \tag{3.5}$$
$$= P(C)/P(I).$$

It is $\bar{\alpha}$ times more likely for a randomly selected pair of members to have consistent order than to have inconsistent order. Whereas $0 \le \bar{\delta} \le 1$, we have $1 \le \bar{\alpha} \le \infty$ with $\bar{\alpha} = 1$ iff $\bar{\delta} = 0$ and $\bar{\alpha} = \infty$ if (but not only if) $\bar{\delta} = 1$. If it is preferred to use a measure having range [0, 1], one obvious alternative is the inverse of $\bar{\alpha}$. Also, we can express $\bar{\alpha}$ as $\bar{\alpha} = (1 + \bar{\gamma})/(1 - \bar{\gamma})$, where $\bar{\gamma} = [P(C) - P(I)]/[P(C) + P(I)]$ is the difference in the proportion of consistent pairs and the proportion of inconsistent pairs, out of those pairs untied on *both* variables. Note that $\bar{\alpha}$ may attain its upper limit for any values of r and c, whereas it is impossible for $\bar{\delta} = 1$ when $r > c$.

3.3 Use With Stochastic Orderings

The measures $\bar{\delta}$ and $\bar{\alpha}$ are most meaningful when the levels of X are stochastically ordered on the ordinal variable. Suppose that level i of X is stochastically larger on Y than level k of X; that is, $\sum_{l=1}^{j} \tilde{\rho}_{il} \le \sum_{l=1}^{j} \tilde{\rho}_{kl}$ for $1 \le j \le c$. Then it may be shown that $\delta_{ik} \ge 0$ and $\ln \alpha_{ik} \ge 0$, and for any level m of X, $\delta_{im} \ge \delta_{km}$ and $\alpha_{im} \ge \alpha_{km}$. If all r levels of X are stochastically ordered on Y, it follows that a labeling of these levels exists for which $\delta_{ik} \ge 0$ whenever $i \ge k$. If the nominal variable X were instead ordinal and had levels ordered from low to high according to this labeling, a pair having consistent order would be concordant and a pair having inconsistent order would be discordant. In that case $\bar{\delta}$ corresponds to Somers's d with X as the independent variable, and $\bar{\alpha}$ is related to gamma as in the $2 \times c$ case.

3.4 Other Approaches

Although our formulation of $\bar{\delta}$ as a difference between two probabilities seems to be original, we note that Freeman (1965, p. 112) defined a related sample measure, called theta (see also Freeman 1976 and Hubert 1974). Additional approaches to measuring nominal-ordinal association may be found in Agresti (1978), Crittenden and Montgomery (1980), Goodman and Kruskal (1959, Secs. 4.4 and 4.5), Jacobson (1972), Řehák (1976), and Sarndal (1974).

4. MODELS FOR NOMINAL-ORDINAL DATA

In the past decade, several authors have proposed models for describing patterns of associations in nominal-ordinal cross-classification tables. These include Andrich (1979), Bock (1975, pp. 541–550), Duncan (1979), Fienberg (1977, pp. 52–58), Goodman (1979), Haberman (1974), McCullagh (1979, 1980) and Williams and Grizzle (1972). In this section we briefly describe the two most commonly considered model types, logit and loglinear. We then discuss how the measures studied in this paper, which are not model based, are appropriate for use in diverse situations described by these and many other models.

4.1 A Logit Model

Let $F_{ij} = \sum_{l=1}^{j} \tilde{\rho}_{il}$ be the jth cumulative probability for category i of the nominal variable, $1 \le i \le r$, $1 \le j \le c$. One way we can use the ordinal nature of the column classification without resorting to scoring methods is by constructing a model, using the accumulated logits $\ln[F_{ij}/(1 - F_{ij})]$, $j = 1, 2, \ldots, c - 1$ within each row. The most popular model of this type is the unsaturated one in which the difference between the distributions for any pair of rows is constant across the columns on this logit scale. That is,

$$\ln[F_{ij}/(1 - F_{ij})] = \ln[F_{kj}/(1 - F_{kj})] + \Delta_{ik}, \quad j = 1, 2, \ldots, c - 1, \qquad (4.1)$$

for all pairs $1 \le i \le k \le r$. This model has been suggested by several authors, including Clayton (1974), Simon (1974), McCullagh (1979, 1980), and Williams and Grizzle (1972). It may equivalently be described by noting that the odds ratios

$$\frac{(\tilde{\rho}_{i1} + \cdots + \tilde{\rho}_{ij})/(\tilde{\rho}_{i,j+1} + \cdots + \tilde{\rho}_{ic})}{(\tilde{\rho}_{k1} + \cdots + \tilde{\rho}_{kj})/(\tilde{\rho}_{k,j+1} + \cdots \tilde{\rho}_{kc})} = \exp(\Delta_{ik}), \quad 1 \le j \le c - 1 \qquad (4.2)$$

are identical for all collapsings of each $2 \times c$ subtable into a 2×2 table. Another formulation of the model, given by Simon (1974), is

$$\ln[F_{ij}/(1 - F_{ij})] = \alpha_i + \beta_j. \qquad (4.3)$$

The logistic differences are $\{\Delta_{ik} = \alpha_i - \alpha_k\}$ in this parameterization.

Although model (4.1) is intuitively appealing, it is not always suitable, even when there are "nice" underlying distributions differing only in location. Fleiss (1970) showed, for example, that for two normal distributions with equal variances, the value of the odds ratio computed for a collapsing into a 2×2 table depends greatly on how the cutting point is chosen for forming the dichotomy. McCullagh (1980) suggested alternative models that also do not require scores. These models assume constant differences $\{\Delta_{ik}\}$ between distribution functions or their complements on a log-log scale.

4.2 A Loglinear Model

This model assumes that a meaningful set of ordered scores $\{V_j\}$ can be assigned to the columns. It contains an interaction term representing a deviation from independence that changes linearly within each row; specifically,

$$\ln \rho_{ij} = \gamma_i + \beta_j + \alpha_i V_j, \quad 1 \le i \le r,\ 1 \le j \le c. \qquad (4.4)$$

Various formulations of this model have been suggested by Simon (1974), Haberman (1974), Goodman (1979), Duncan (1979), and Fienberg (1977, pp. 52–55).

An interesting implication of model (4.4) is that

$$\ln(\rho_{ij}\rho_{kl}/\rho_{il}\rho_{kj}) = (\alpha_i - \alpha_k)(V_j - V_l). \qquad (4.5)$$

In other words, for any pair of rows, the odds ratio is constant for all pairs of columns that are equidistant in score. For the equal-interval scores $\{V_j = j\}$, the log odds ratio equals $\Delta_{ik} = \alpha_i - \alpha_k$ for all pairs of adjacent columns. Thus, any pair of rows has a constant difference between log odds of adjacent cells proportions,

$$\Delta_{ik} = \ln(\rho_{ij}/\rho_{i,j+1}) - \ln(\rho_{kj}/\rho_{k,j+1}), \quad 1 \le j \le c - 1. \qquad (4.6)$$

4.3 Measuring Association

When model (4.1) or (4.4) provides an adequate fit to a table, the strength of the association can be quantified by the magnitude of the $\{\Delta_{ik}\}$ "difference" parameters. (Notice that $r - 1$ of these parameters (and hence the $\{\alpha_i\}$) determine the entire set of $\{\Delta_{ik}\}$.) Since the meaning of Δ_{ik} depends on the particular structural form for the model, however, the magnitudes of these parameters cannot be compared across differing structural models. This makes it difficult to compare associations in two or more tables for which different structural models are applicable. In addition, many cross-classifications occur in practice for which (a) none of the commonly used structural models provides an adequate fit, or (b) if a good-fitting model is obtained by trying several structural types, the result of "fishing for structure" may be that the same model type is inadequate when applied to other cross-classifications of the same variables.

The delta and alpha measures of nominal-ordinal association, not being model based, can often be used for comparing strengths of association across tables even if no single, simple structural model form is generally applicable to those tables. These measures have a certain robustness in the sense that they are applicable in a broad range of settings that would encompass a variety of models. We showed in Section 3.3 that they are naturally suited to systems of stochastically ordered distributions. Now it can be seen that any cross-classification table for which the logistic model (4.1), loglinear model (4.4), or one of McCullagh's (1980) log-log models fits perfectly is such that the distributions within the rows are stochastically ordered. Thus, the delta and alpha measures are suitable for use whenever one of these important model types is deemed appropriate. Their robustness is illustrated by the fact that if *any* of these models fits a particular table perfectly, then the $\{\Delta_{ij}\}$ for that model will be matched in sign by the $\{\delta_{ij}\}$ and $\{\ln\alpha_{ij}\}$. When different model types fit different tables, these measures give us a common basis for comparing associations and summarizing the results of the models.

Table 1, introduced earlier, illustrates the above points. The loglinear model (4.4) provides a poor fit to these data. Goodness-of-fit tests are of little interest for these estimated population frequencies. Nevertheless, the likelihood ratio chi-squared statistic $G^2 = 2.08 \times 10^6$, based on df $= 1$, is large even for the size of this data set. Closer inspection reveals that the two differences between log odds of adjacent cell frequencies are quite different and even have different signs. The logit model provides a similar fit and also is inadequate, with $G^2 = 2.14 \times 10^6$ based on df $= 1$. In fact, note that the magnitudes of the estimated expected frequencies in each row for these models even differ in order from the observed frequencies (e.g., for the North row, the largest expected frequency occurs in the cell with the smallest observed frequency). Poor fits are also obtained with other unsaturated models we have considered, such as the log-log models.

The distributions in the two rows of Table 1 are stochastically ordered, however, so δ and α provide meaningful summaries. The tendency for people in the North to be more highly metropolitan is reflected by the simply interpretable values $\hat{\delta} = .151$ and $\hat{\alpha} = 1.574$. For example, $\hat{\alpha} = 1.574$ means that for a randomly selected pair from Table 1 (one observation from each row), it is 1.574 times as likely that the member from the North lives in the more highly metropolitan area than it is that the member from the South lives in the more highly metropolitan area.

These remarks are strengthened by the fact that similar behavior occurs when data from other years are analyzed and when region is measured with more categories. For example, the loglinear and logit models provide poor fits to the corresponding data from 1960 ($G^2 = 2.11 \times 10^6$ and $G^2 = 2.30 \times 10^6$, respectively). In each year the G^2 values are even larger when four levels (Northeast, North central, South, West) are used for region. However, in each year the regions have the stochastic ordering NE $>$ W $>$ NC $>$ S on size of residential area, so the $\{\delta_{ij}\}$ and $\{\alpha_{ij}\}$ provide simple summaries. Their use also results in interesting and substantive conclusions. For example, all $\{ | \hat{\delta}_{ij} | \}$ and $\{ | \ln\hat{\alpha}_{ij} | \}$ from 1960 exceed the corresponding values from 1975. This indicates that differences between regions in the distribution of size of residential area tended to diminish over these 15 years. This slight decrease in variability among the regions is also reflected by the smaller values in 1975 of the summary measures $\bar{\delta}$ and $\bar{\alpha}$ (.224 and 2.008, respectively, compared with .259 and 2.244 in 1960). Within each region one could also compute δ or α for pairs of years to summarize change toward metropolitan populations.

The above remarks are not intended as a criticism of the model-building approach. It is important to attempt to describe table structures, and we believe that these measures help to complement that process. They describe strength of association on a common basis for the class of tables of stochastically ordered distributions, different elements of which may be well fit by different models or by none of the simple models in current use.

APPENDIX: DERIVATIONS OF ASYMPTOTIC SAMPLING DISTRIBUTIONS

The population values of $\bar{\delta}$ and $\bar{\alpha}$ can be expressed as $\xi = \nu/\Delta$, where ν and Δ are functions of the $\{\rho_{ij}\}$. Let $\hat{\xi}$ denote the sample value of ξ, $\phi_{ij} = \nu(\partial\Delta/\partial\rho_{ij}) - \Delta(\partial\nu/\partial\rho_{ij})$, and $\bar{\phi} = \sum_{i,j}\rho_{ij}\phi_{ij}$. Using the "delta method," Goodman and Kruskal (1972) showed that for full multinomial sampling, $\sqrt{n}(\hat{\xi} - \xi)/\sigma_{\xi} \xrightarrow{d} N(0, 1)$ as the sample size $n \rightarrow \infty$, where

$$\sigma_{\hat{\xi}}^2 = \sum_{i,j} \rho_{ij}(\phi_{ij} - \bar{\phi})^2/\Delta^4. \tag{A.1}$$

For independent multinomial sampling with the $\{\rho_{i\cdot}\}$ known and with sampling proportions $\{w_i\}$, the same asymptotic distribution occurs but with

$$\sigma_{\hat{\xi}}^2 = \sum_i \frac{1}{w_i} \sum_j \tilde{\rho}_{ij}(\phi_{ij}{}^+ - \bar{\phi}_i{}^+)^2/\Delta^4, \tag{A.2}$$

where $\phi_{ij}{}^+ = \nu(\partial\Delta/\partial\tilde{\rho}_{ij}) - \Delta(\partial\nu/\partial\tilde{\rho}_{ij})$ and $\bar{\phi}_i{}^+ = \sum_j \tilde{\rho}_{ij}\phi_{ij}{}^+$. Substitution of the sample proportions into either asymptotic variance formula yields a consistent estimate $\hat{\sigma}_{\hat{\xi}}{}^2$ of $\sigma_{\hat{\xi}}{}^2$, which can be used in constructing confidence intervals for ξ. In this Appendix we give the expressions for ϕ_{ij} and $\phi_{ij}{}^+$ to be inserted into (A.1) and (A.2) for the cases $\xi = \bar{\delta}$ and $\xi = \bar{\alpha}$.

We first consider the asymptotic distribution of the sample analog $\hat{\delta}$ of δ for full multinomial sampling. The consistency of all sample cell proportions implies that all $\hat{\delta}_{ik} \xrightarrow{P} \delta_{ik}$. We assume that all $\delta_{ik} \neq 0$, which implies that

$$P(\hat{G}_i{}^+ \equiv G_i{}^+ \quad \text{and} \quad \hat{G}_i{}^- \equiv G_i{}^-, \quad \text{all} \quad i) \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty. \tag{A.3}$$

It follows from a lemma in Goodman and Kruskal (1963, p. 357) that for asymptotic purposes we may treat $\hat{G}_i{}^+ \equiv G_i{}^+$ and $\hat{G}_i{}^- \equiv G_i{}^-$. Now, letting $\bar{\delta} = \nu/\Delta$ with $\Delta = 2\sum_{i<k}\rho_{i\cdot}\rho_{k\cdot}$, we obtain $\phi_{ij} = 2\nu(1 - \rho_{i\cdot}) - 2\Delta(R_{ij}{}^{(C)}$

$- R_{ij}^{(I)})$ and $\bar{\phi} = 0$. If $\bar{\delta} = 1$, then $R_{ij}^{(C)} = 1 - \rho_{i\cdot}$ and $R_{ij}^{(I)} = 0$ all i, j, so that $\sigma_{\hat{\delta}}^2 = 0$; then $\hat{\delta} = 1$ and $\hat{\sigma}_{\hat{\delta}}^2 = 0$ with probability one.

In applications in which we are comparing several groups on an ordinal response, the sampling scheme is often independent multinomial within the levels of X. In that case we assume the $\{\rho_{i\cdot}\}$ are known, and we obtain $\phi_{ij}^+ = -2\rho_{i\cdot}\Delta(R_{ij}^{(C)} - R_{ij}^{(I)})$ and $\bar{\phi}_i^+ = -2\Delta\sum\rho_{ij}(R_{ij}^{(C)} - R_{ij}^{(I)})$.

Next we consider the sample version $\hat{\alpha}$ of $\bar{\alpha}$, under the assumption that all $\alpha_{ij} \neq 1$. Letting $\bar{\alpha} = \nu/\Delta$ with $\Delta = P(I)$, we obtain $\phi_{ij} = 2\nu R_{ij}^{(I)} - 2\Delta R_{ij}^{(C)}$ and $\bar{\phi} = 0$ for the case of full multinomial sampling. For the case of independent multinomial sampling with known $\{\rho_{i\cdot}\}$, we obtain $\phi_{ij}^+ = 2\nu\rho_{i\cdot}R_{ij}^{(I)} - 2\Delta\rho_{i\cdot}R_{ij}^{(C)}$ and $\bar{\phi}_i^+ = 2\nu\sum_j\rho_{ij}R_{ij}^{(I)} - 2\Delta\sum_j\rho_{ij}R_{ij}^{(C)}$.

Being a difference rather than a ratio, $\ln(\hat{\alpha})$ tends to converge faster to its limiting normal distribution. Its variance can be estimated for large samples by $\hat{\sigma}_{\hat{\alpha}}^2/n\hat{\alpha}^2$. Thus, one can form the $100(1 - p)$percent confidence interval $\ln(\hat{\alpha}) \pm z_{p/2}\hat{\sigma}_{\hat{\alpha}}\sqrt{n}\hat{\sigma}$ for $\ln(\bar{\alpha})$ and then exponentiate endpoints to obtain a corresponding confidence interval for $\bar{\alpha}$.

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