STA6329 Practice Problem Solutions

(1) $TC = \underset{\sim}{c}'A\underset{\sim}{b} = [c_1\ c_2\ c_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34}\end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4\end{bmatrix}$

$= [c_1a_{11}+c_2a_{21}+c_3a_{31} \vdots c_1a_{12}+c_2a_{22}+c_3a_{32} \vdots c_1a_{13}+c_2a_{23}+c_3a_{33} \vdots c_1a_{14}+c_2a_{24}+c_3a_{34}]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4\end{bmatrix}$

$= b_1(c_1a_{11}+c_2a_{21}+c_3a_{31}) + b_2(c_1a_{12}+c_2a_{22}+c_3a_{32}) + b_3(c_1a_{13}+c_2a_{23}+c_3a_{33}) + b_4(c_1a_{14}+c_2a_{24}+c_3a_{34})]$

Price to charge: $1.10\,TC$

$TC = [100\ \ 200\ \ 150]\begin{bmatrix} 1.19 \\ 1.23 \\ 1.25\end{bmatrix} = 119 + 246 + 187.50 = 552.5$

$1.1 \times TC = 1.1(552.5) = 607.75.$

(2) a) ~~(2x3)x(3x3)~~ $m \times p$ b) 2 rows x 3 columns (Note B has "reversed" subscripts)

$$AB = \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21}+A_{13}B_{31} & A_{11}B_{12}+A_{12}B_{22}+A_{13}B_{32} & A_{11}B_{13}+A_{12}B_{23}+A_{13}B_{33} \\ A_{21}B_{11}+A_{22}B_{21}+A_{23}B_{31} & A_{21}B_{12}+A_{22}B_{22}+A_{31}B_{32} & A_{21}B_{13}+A_{22}B_{23}+A_{23}B_{33}\end{bmatrix}$$

(3) $U_{ij} = \underset{\sim}{e}_i\underset{\sim}{u}_j' = \begin{cases} 1 & \text{if row} = i,\ \text{col} = j \\ 0 & \text{ow.}\end{cases}$ $\quad A = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\underset{\sim}{e}_i\underset{\sim}{u}_j'$

(4) $\sum_{i=1}^{k} x_iA_i = 0 \Rightarrow -x_iA_i = \sum_{\substack{j=1\\j\neq i}}^{k} x_jA_j \Rightarrow A_i = \sum_{\substack{j=1\\j\neq i}}^{k} -\frac{x_j}{x_i}A_j$

(5) $k=0$: $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow$ linearly dependent $(x_1 = -x_3,\ x_2 = 0)$

In general: $x_1k + x_2 = 0 \Rightarrow x_1 = -\frac{x_2}{k} \qquad x_2 + x_3k = 0 \Rightarrow x_3 = -\frac{x_2}{k}$

$x_1 + x_2k + x_3 = -\frac{x_2}{k} + x_2k - \frac{x_2}{k} = 0 \Rightarrow x_2k = \frac{2x_2}{k} \Rightarrow k^2 = 2 \Rightarrow k = \pm\sqrt{2}$

(6) $x_1 A + x_2 B + x_3 C = 0 \Rightarrow x_1 = x_2 = x_3 = 0$

$C_j = x_{1j} A + x_{2j} B + x_{3j} C \Rightarrow \underset{\sim}{x}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ $\underset{\sim}{x}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\underset{\sim}{x}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$

$\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3$ are linearly independent $\Rightarrow A+B, A+C, B+C \equiv$ lin. indep. (Lem 3.2.4)

(7) $A_1 \in \mathcal{V}$, $A_2 \in \mathcal{V} \Rightarrow A_1 + A_2 \in \mathcal{V}$ (sum of even #'s are even)

If $k \equiv$ integer valued, $kA_1 \in \mathcal{V}$, however k must be any scalar

$\Rightarrow$ No, not a linear space (only integers can be even/odd)

(8) a), b) $x_1 \begin{bmatrix}1\\-7\\3\end{bmatrix} + x_2 \begin{bmatrix}0\\2\\3\end{bmatrix} + x_3 \begin{bmatrix}5\\1\\3\end{bmatrix} = \begin{bmatrix}25\\-139\\63\end{bmatrix}$ yes $x_1 = 20, x_2 = 0, x_3 = 1$

c) $x_1 [1\ 0\ 5] + x_2 [-7\ 2\ 1] + x_3 [3\ 3\ 3]$

d) $2x_2 + 3x_3 = 0 \Rightarrow x_2 = -\frac{3}{2} x_3 \Rightarrow 1x_1 - 7(-\frac{3}{2}x_3) + 3x_3 = 0 \Rightarrow x_1 = -\frac{27}{2} x_3$

$\Rightarrow 5\left(\frac{-21}{2} x_3\right) + 1\left(-\frac{3}{2} x_3\right) + 3x_3 = 51 x_3 = 0 \Rightarrow x_3 = 0 = x_1 = x_2$

No $[0\ 0\ 0]$ is not in row space of $\underline{A}$

(9) $A = \begin{bmatrix}1 & 1\\1 & 2\\1 & 3\end{bmatrix}$ $B = \begin{bmatrix}5\\9\\13\end{bmatrix}$ $C = \begin{bmatrix}-5 & -5\\-5 & -10\\-5 & -15\end{bmatrix}$

$\mathcal{C}(A) < \mathcal{C}(B) \Rightarrow A = BF$ for some matrix F No, $\mathcal{C}(A)$ not $\subset \mathcal{C}(B)$

$\mathcal{C}(B) < \mathcal{C}(A) \Rightarrow B = AF$ Let $F = \begin{bmatrix}1\\4\end{bmatrix}$, then $AF = B$ Yes

$\mathcal{R}(A) \subset \mathcal{R}(C) \Rightarrow LA = C$ for some L Let ~~$L = [-5\ -5\ -5]$~~,

Let $L_A = -5 \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ Then $L_A A = C$

$\mathcal{R}(C) \subset \mathcal{R}(A) = LC = A$ for some L Let $L_C = -\frac{1}{5}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

$\Rightarrow L_C C = A \Rightarrow \mathcal{R}(A) = \mathcal{R}(C)$

(10) a) $-5\underset{\sim}{c}_{A_1} + 1\underset{\sim}{c}_{A_2} - 1\underset{\sim}{c}_{A_3} = \underset{\sim}{0} \Rightarrow \text{rank}(A) = 2$

b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ c) $\left\{ \begin{matrix} [1 & 1 & -4] \\ [1 & 2 & -3] \end{matrix} \right\}$

d)

$x_1 + x_2 + x_3 = 0 \quad (1)$

$x_1 + 2x_2 + 3x_3 = 0 \quad (2)$

$-4x_1 - 3x_2 + 2x_3 = 0 \quad (3)$

$(2)-(1) \Rightarrow x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_3$

$(1) \Rightarrow x_1 - 2x_3 + x_3 = 0 \Rightarrow x_1 = x_3$

Let $x_3 = 1 \Rightarrow x_2 = -2,\ x_1 = 1$

$1[1 \ 1 \ -4] - 2[1 \ 2 \ -3] + 1[1 \ 3 \ -2] = [0 \ 0 \ 0]$ ✓

e) $[1 \ 3 \ -2] = -1[1 \ 1 \ -4] + 2[1 \ 2 \ -3]$

(11) a) $\mathcal{C}(A) \subset \mathcal{C}(A,B) \ni A = [A \ B]\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ Let $F = \begin{bmatrix} I \\ 0 \end{bmatrix}$ ✓

$\mathcal{R}(A) \subset \mathcal{R}\begin{pmatrix} A \\ C \end{pmatrix} \Rightarrow A = [L_1 \ L_2]\begin{bmatrix} A \\ C \end{bmatrix}$ Let $L = [I \ 0]$

b) $\mathcal{C}(B) \subset \mathcal{C}(A) \Rightarrow B = AF$

$\Rightarrow [A \ B] = [AI \ AF] = A[I \ F] = AF^*$

$\Rightarrow \mathcal{C}([A \ B]) \subset \mathcal{C}(A) \Rightarrow \mathcal{C}(A) = \mathcal{C}(A,B)$

(12) $\text{rank}(A) = 3$, $\text{rank}(B) = 2$, $\text{rank}(C) = 2$

$\text{rank}(A \ B) = \text{rank}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = 3 \qquad 3 \leq 3 + 2 = 5$

$\text{rank}\begin{pmatrix} A \\ C \end{pmatrix} = \text{rank}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} = 3 \qquad 3 \leq 3 + 2 = 5$

(13) $AB_{11} = \sum_{k=1}^{n} a_{1k} b_{k1}$ $AB_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$ $tr(AB) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} b_{ki}$

$BA_{11} = \sum_{k=1}^{m} b_{1k} a_{k1}$ $BA_{ii} = \sum_{k=1}^{m} b_{ik} a_{ki}$ $tr(BA) = \sum_{i=1}^{n} \sum_{k=1}^{m} b_{ik} a_{ki}$

$$= \sum_{k=1}^{m} \sum_{i=1}^{n} a_{ki} b_{ik} = tr(AB)$$

(14) $(A'A)_{ii} = \sum_{j=1}^{n} a_{ij}^2 \Rightarrow tr(A'A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$

if $A = 0$ then $A'A = 0 \Rightarrow tr(A'A) = 0$

if $tr(A'A) = 0$, then $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 = 0 \Rightarrow a_{ij} = 0 \ \forall i, j$

if A is a real-valued matrix.

(15) $AB = AC \Rightarrow A'(AB) = A'(AC) \Rightarrow A'AB = A'AC$

$A'AB = A'AC \Rightarrow A'(AB - AC) = 0 \Rightarrow (B-C)'A'(AB - AC) = 0$

$\Rightarrow (AB - AC)'(AB - AC) = 0 \Rightarrow AB - AC = 0$

(16) a) $\|x\| = \sqrt{x_1^2 + x_2^2} = \sqrt{25+4} = \sqrt{29}$

b) $\|y\| = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$

c) $d(x, y) = \sqrt{(5 - -2)^2 + (2-4)^2} = \sqrt{49+4} = \sqrt{53}$

d) $x \cdot y = 5(-2) + 2(4) = -10 + 8 = -2$

$$\cos\theta = \frac{-2}{\sqrt{29}\sqrt{20}} = \frac{-2}{24.0832} = -.0830 \Rightarrow \theta = \cos^{-1}(-.0830)$$

$$= 94.76^\circ \ (1.654 \text{ rads})$$

17 a) $A \cdot B = tr(A'B) = tr\left[\begin{bmatrix} 4 & 3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}\right] = tr\begin{bmatrix} 5 & 11 \\ 23 & 3 \end{bmatrix} = 8$

b) $\|A\| = \sqrt{\sum_i \sum_j a_{ij}^2} = \sqrt{16+4+9+49} = \sqrt{78} = 8.832$

c) $\|B\| = \sqrt{1+4+9+1} = \sqrt{15} = 3.873$

d) $\delta(A,B) = \|A-B\| = \left\| \begin{matrix} 5 & -4 \\ 0 & 6 \end{matrix} \right\| = \sqrt{25+16+0+36} = \sqrt{77} = 8.775$

$\cos\theta = \dfrac{8}{8.832(3.873)} = 0.234$

18 $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \Rightarrow A'B = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \Rightarrow tr(A'B) = 0$

19 $\underset{\sim}{b}_1 = \underset{\sim}{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ $x_{12} = \dfrac{\underset{\sim}{a}_2 \cdot \underset{\sim}{b}_1}{\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_1} = \dfrac{2+12-2+2}{1+9+4+1} = \dfrac{14}{15}$

$\underset{\sim}{b}_2 = \underset{\sim}{a}_2 - x_{12}\underset{\sim}{b}_1 = \begin{bmatrix} 2 \\ 4 \\ -1 \\ 2 \end{bmatrix} - \dfrac{14}{15}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16/15 \\ 18/15 \\ -43/15 \\ 16/15 \end{bmatrix}$

$x_{13} = \dfrac{\underset{\sim}{a}_3 \cdot \underset{\sim}{b}_1}{\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_1} = \dfrac{3+6+0+1}{15} = \dfrac{13}{15}$

$x_{23} = \dfrac{\underset{\sim}{a}_3 \cdot \underset{\sim}{b}_2}{\underset{\sim}{b}_2 \cdot \underset{\sim}{b}_2} = \dfrac{\frac{1}{15}[48+36+0+16]}{(\frac{1}{15})^2[256+324+1849+256]} = \dfrac{15(100)}{2685} = \dfrac{100}{179}$

$\Rightarrow \underset{\sim}{b}_3 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \dfrac{100}{179}\begin{bmatrix} 16/15 \\ 18/15 \\ -43/15 \\ 16/15 \end{bmatrix} - \dfrac{13}{15}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

$= \dfrac{1}{2685}\begin{bmatrix} 3(2685) - 100(16) - 13(179)(1) \\ 2(2685) - 100(18) - 13(179)(3) \\ 0 - 100(-43) - 13(179)(2) \\ 1(2685) - 100(16) - 13(179)(1) \end{bmatrix} = \dfrac{1}{2685}\begin{bmatrix} 4128 \\ -3411 \\ -354 \\ -1242 \end{bmatrix}$

(19) continued

$$B = \frac{1}{2685}\begin{bmatrix} 2685 & 2864 & 4128 \\ 8055 & 3222 & -3411 \\ 5370 & -7697 & -354 \\ 2685 & 2864 & -1242 \end{bmatrix}$$

$\|b_1\| = \sqrt{1+9+4+1} = \sqrt{15} = 3.87298$ $\quad \|b_2\| = \sqrt{\frac{2685}{15^2}} = 3.45447$

$$\|b_3\| = \sqrt{\frac{4128^2+3411^2+354^2+1242^2}{2685^2}} = \sqrt{\frac{1542564}{2685^2}} = .46257$$

$$D = \begin{bmatrix} \frac{1}{3.87298} & 0 & 0 \\ 0 & \frac{1}{3.45447} & 0 \\ 0 & 0 & \frac{1}{.46257} \end{bmatrix}$$

$Q =$

$$BD = \frac{1}{2685}\begin{bmatrix} 693.2646 & 829.0707 & 8924.0547 \\ 2079.7939 & 932.7046 & -7374.0191 \\ 1386.5292 & -2228.1276 & -765.2896 \\ 693.2646 & 829.0707 & -2684.9990 \end{bmatrix}$$

$$\underset{k\times k}{X} = \begin{bmatrix} 1 & 14/15 & 13/15 \\ 0 & 1 & \frac{100}{179} \\ 0 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 3.87298 & 0 & 0 \\ 0 & 3.45447 & 0 \\ 0 & 0 & .46257 \end{bmatrix}$$

$$R = EX = \begin{bmatrix} 3.87298 & 3.61478 & 3.35658 \\ 0 & 3.45447 & \cancel{0.255842}\ 1.93000 \\ 0 & 0 & 0.46257 \end{bmatrix}$$

$$A = QR = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

STA 6329 - Practice Prob Sols.

20 a) $AX = \begin{bmatrix} 7 & 3 \\ 21 & 12 \\ 32 & 16 \\ 22 & 10 \\ 8 & 4 \\ 9 & 4 \\ 99 & 49 \end{bmatrix} = B$

b) $\mathcal{C}(B) \subset \mathcal{C}(A)$

$\Rightarrow B = AF$ for some F

$B = A \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = AF$

c) $K_1'A = [\overbrace{0+3+3+1+0+0-7}^{7} ; \overbrace{2+9+10+6+4+3-34}^{34} : \overbrace{1+0+3+3+0+1-8}^{8}]$

$= [0 \quad 0 \quad 0]$

$K_1'B = [\overbrace{7+21+32+22+8+9-99}^{99} ; \overbrace{3+12+16+10+4+4-49}^{49}] = (0 \quad 0)$

$K_2'A : [\overbrace{2.4(0)+0.8(3)-0.8(3)+0(1)-1(0)+0+0}^{0} ;$

$\underbrace{2.4(2)+0.8(9)-.8(10)-1(4)}_{0} ; \underbrace{2.4(1)+0.8(0)-0.8(3)-1(0)}_{0}]$

$K_2'B = [\underbrace{2.4(7)+0.8(21)-0.8(32)-1(8)}_{16.8+16.8-25.6-8=0} ; \underbrace{2.4(3)+0.8(12)-0.8(16)-1(4)}_{7.2+9.6-12.8-4=0}]$

21

$x_1 + 14x_2 = 79 \Rightarrow x_1 = 79 - 14x_2$

$x_1 + 29x_2 = 57 \Rightarrow 79 - 14x_2 + 29x_2 = 57 \Rightarrow 15x_2 = -22 \Rightarrow x_2 = \frac{-22}{15}$

$x_1 + 38x_2 = 28$

$\Rightarrow x_1 = 79 - 14\left(\frac{-22}{15}\right) = 79 + \frac{308}{15} = \frac{1493}{15}$

$\frac{1493 + 38(-22)}{15} = \frac{1493 - 836}{15} = \frac{657}{15} = 43.8 \neq 28$

(21) Continued b) $A'A = \begin{bmatrix} 3 & 81 \\ 81 & 2481 \end{bmatrix}$ $A'B = \begin{bmatrix} 164 \\ 3823 \end{bmatrix}$

$3x_1 + 81x_2 = 164 \Rightarrow x_1 = \frac{1}{3}(164 - 81x_2) = \frac{164}{3} - 27x_2$

$81\left(\frac{164}{3} - 27x_2\right) + 2481x_2 = 3823 \Rightarrow 4428 + 294x_2 = 3823$

$\Rightarrow 605 = -294x_2 \Rightarrow x_2 = -2.0578$

$\Rightarrow x_1 = \frac{164}{3} - 27(-2.0578) = 110.2273$