Section 14.1 - Terminology and Basic Results

$\underset{n\times 1}{\underset{\sim}{a}} = [a_1 \cdots a_n]' \equiv$ arbitrary $n\times 1$ column vector.

Function that assigns to each $\underset{\sim}{x} \in \mathbb{R}^n$ the value $\underset{\sim}{a}'\underset{\sim}{x} = \sum_{i=1}^{n} a_i x_i \equiv$ linear form in $\underset{\sim}{x}$

Linear form $\equiv$ homogeneous polynomial of degree 1. $\underset{\sim}{a}'\underset{\sim}{x} = \underset{\sim}{x}'\underset{\sim}{a}$

$a_1, \dots, a_n \equiv$ Coefficients of the linear form $\underset{\sim}{a}', \underset{\sim}{a} \equiv$ coefficient vector

Function $f(\underset{\sim}{x})$ of $\underset{\sim}{x}$ (w/ domain $\mathbb{R}^n$) is said to be linear if:

(1) $f(\underset{\sim}{x}_1 + \underset{\sim}{x}_2) = f(\underset{\sim}{x}_1) + f(\underset{\sim}{x}_2) \quad \forall\ \underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathbb{R}^n$

(2) $f(c\underset{\sim}{x}) = c f(\underset{\sim}{x}) \quad \forall$ scalar c and vector $\underset{\sim}{x} \in \mathbb{R}^n$

(3) In general: $f(c_1\underset{\sim}{x}_1 + \dots + c_k\underset{\sim}{x}_k) = c_1 f(\underset{\sim}{x}_1) + \dots + c_k f(\underset{\sim}{x}_k)$

Linear form $\underset{\sim}{a}'\underset{\sim}{x} = \sum_i a_i x_i \equiv$ linear function of $\underset{\sim}{x} = [x_1 \cdots x_n]'$

Conversely, $f(\underset{\sim}{x}) \equiv$ expressible as linear form in $\underset{\sim}{x}$

Take $a_i = f(\underset{\sim}{e}_i)$ $\quad i=1,\dots,n$ where $I \equiv [\underset{\sim}{e}_1 \cdots \underset{\sim}{e}_n]$

$\Rightarrow f(\underset{\sim}{x}) = f(x_1\underset{\sim}{e}_1 + \dots + x_n\underset{\sim}{e}_n) = x_1 f(\underset{\sim}{e}_1) + \dots + x_n f(\underset{\sim}{e}_n) = a_1x_1 + \dots + a_nx_n = \underset{\sim}{a}'\underset{\sim}{x}$

If coefficient vectors $\underset{\sim}{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ of two linear forms $\underset{\sim}{a}'\underset{\sim}{x}$, $\underset{\sim}{b}'\underset{\sim}{x}$ are equal, then the two linear forms are identically equal (equal $\forall\ \underset{\sim}{x}$)

If two linear forms $\underset{\sim}{a}'\underset{\sim}{x}$ and $\underset{\sim}{b}'\underset{\sim}{x}$ are identically equal, then $\underset{\sim}{a} = \underset{\sim}{b}$
(Let $\underset{\sim}{x} \equiv$ columns of I_n)

Let $\underset{m\times n}{A} = \{a_{ij}\}$, $\underset{m\times 1}{\underset{\sim}{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$, $\underset{n\times 1}{\underset{\sim}{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$, then the function:

$$\underset{\sim}{x}'A\underset{\sim}{y} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} x_i y_j \equiv \text{bilinear form}$$

Bilinear form $\equiv$ special type of 2^{nd} degree homogeneous polynomial in $x_1, \dots, x_m, y_1, \dots, y_n$

which: 1) for fixed $\underset{\sim}{x} \equiv$ homogeneous 1^{st} degree polynomial in $y_1, \dots, y_n$

2) " " $\underset{\sim}{y} \equiv$ " " " " " $x_1, \dots, x_m$

$A \equiv$ matrix of the Bilinear form $\underset{\sim}{x}' A \underset{\sim}{y}$

Note that $\underset{\sim}{x}' A \underset{\sim}{y} = \underset{\sim}{y}' A' \underset{\sim}{x} \equiv$ Bilinear form in $\underset{\sim}{y}, \underset{\sim}{x}$ w/ coeff. matrix A'

A function $f(\underset{\sim}{x}, \underset{\sim}{y})$ of $\underset{\sim}{x}, \underset{\sim}{y}$ is said to be <u>bilinear</u> if, $\forall$ fixed $\underset{\sim}{x}$ it's a linear function of $\underset{\sim}{y}$, and vice versa.

$\forall$ fixed $\underset{\sim}{y}$, $\underset{\sim}{x}' A \underset{\sim}{y} = \underset{\sim}{x}' \underset{\sim}{a}^* \equiv$ linear form in $\underset{\sim}{x}$

$\forall$ fixed $\underset{\sim}{x}$ $\underset{\sim}{x}' A \underset{\sim}{y} = \underset{\sim}{b}^{*\prime} \underset{\sim}{y} \equiv$ " " " $\underset{\sim}{y}$

Bilinear form $\equiv$ Bilinear function, Bilinear function can be written in bilinear form

Let $I_m = [\underset{\sim}{e}_1, \dots, \underset{\sim}{e}_m]$ $I_n = [\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_n]$ $\underset{m \times n}{A} = \{a_{ij}\}$ w/ $a_{ij} = f(\underset{\sim}{e}_i, \underset{\sim}{u}_j)$

$\Rightarrow f(\underset{\sim}{x}, \underset{\sim}{y}) = f(\sum_i x_i \underset{\sim}{e}_i, \underset{\sim}{y}) = \sum_i x_i f(\underset{\sim}{e}_i, \underset{\sim}{y}) = \sum_i x_i f(\underset{\sim}{e}_i, \sum_j y_j \underset{\sim}{u}_j)$

$= \sum_i x_i \sum_j y_j f(\underset{\sim}{e}_i, \underset{\sim}{u}_j) = \sum_i \sum_j a_{ij} x_i y_j = \underset{\sim}{x}' A \underset{\sim}{y}$

If $A = \{a_{ij}\}$ and $B \equiv \{b_{ij}\}$ of two bilinear forms $\underset{\sim}{x}' A \underset{\sim}{y}$, $\underset{\sim}{x}' B \underset{\sim}{y}$ in $\underset{\sim}{x}, \underset{\sim}{y}$ are equal then the two bilinear forms are identically equal (equal $\forall \underset{\sim}{x}, \underset{\sim}{y}$)

Conversely if two bilinear forms are identically equal, then $A = B$

<u>Proof</u>: Let $\underset{\sim}{x} = \underset{\sim}{e}_i$, $\underset{\sim}{y} = \underset{\sim}{u}_j$ $\Rightarrow a_{ij} = \underset{\sim}{x}' A \underset{\sim}{y} = \underset{\sim}{x}' B \underset{\sim}{y} = b_{ij}$ $\begin{matrix} i = 1, \dots, m \\ j = 1, \dots, n \end{matrix}$

A bilinear form $\underset{\sim}{x}' A \underset{\sim}{y}$ in $\underset{n \times 1}{\underset{\sim}{x}}, \underset{n \times 1}{\underset{\sim}{y}} \in \mathbb{R}^n$ is symmetric if $\underset{\sim}{x}' A \underset{\sim}{y} = \underset{\sim}{y}' A \underset{\sim}{x}$ $\forall \underset{\sim}{x}, \underset{\sim}{y}$

$\underset{\sim}{y}' A \underset{\sim}{x} = (\underset{\sim}{y}' A \underset{\sim}{x})' = \underset{\sim}{x}' A' \underset{\sim}{y}$, then $\underset{\sim}{x}' A \underset{\sim}{y} \equiv$ symmetric iff $\underset{\sim}{x}' A \underset{\sim}{y} = \underset{\sim}{x}' A' \underset{\sim}{y}$ $\forall \underset{\sim}{x} \underset{\sim}{y} \Leftrightarrow A = A'$

($\underset{\sim}{e}_i, \underset{\sim}{u}_j$ arguments)

$\Rightarrow$ Bilinear form is symmetric iff the coefficient matrix of the bilinear form is symmetric.

Quadratic forms: $\underset{n\times n}{A} = \{a_{ij}\} \equiv$ arbitrary square matrix

Function that assigns to each $\underset{n\times 1}{\underset{\sim}{x}} \in \mathbb{R}^n$ the value $\underset{\sim}{x}'A\underset{\sim}{x} = \sum_i\sum_j a_{ij} x_i x_j$

$= \sum_i a_{ii} x_i^2 + \sum_{i,\, j\neq i} a_{ij} x_i x_j \equiv$ Quadratic form in $\underset{\sim}{x}$

Quadratic form $\equiv$ homogeneous polynomial of degree 2. $A \equiv$ matrix of quad form $\underset{\sim}{x}'A\underset{\sim}{x}$

.. .. can be obtained from bilinear form w/ $\underset{\sim}{y} = \underset{\sim}{x}$

Let $\underset{n\times n}{B} = \{b_{ij}\}$ circumstances for $\underset{\sim}{x}'A\underset{\sim}{x}$ & $\underset{\sim}{x}'B\underset{\sim}{x} \equiv$ identically equal

Sufficient condition: $A = B$, except for $n=1$, not necessary condition.

suppose $\underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'B\underset{\sim}{x}$ Let $\underset{\sim}{x} = \underset{\sim}{e}_i$ (i^{th} column of I_n) $\Rightarrow a_{ii} = \underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'B\underset{\sim}{x} = b_{ii}$ $i = 1,\dots,n$ (1.1)

Let $\underset{\sim}{x} = \underset{\sim}{e}_i + \underset{\sim}{e}_j \Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} = [a_{i1} + a_{j1} \cdots a_{in} + a_{jn}]\underset{\sim}{x} = a_{ii} + a_{ji} + a_{ij} + a_{jj}$

Similarly, $\underset{\sim}{x}'B\underset{\sim}{x} = b_{ii} + b_{ji} + b_{ij} + b_{jj} = a_{ii} + a_{ji} + a_{ij} + a_{jj}$ (1.2) $j \neq i = 1,\dots,n$

$\Rightarrow b_{ij} + b_{ji} = a_{ij} + a_{ji} \Rightarrow A + A' = B + B'$ (1.3) (necessary, sufficient condition)

$$\underset{\sim}{x}'A\underset{\sim}{x} = \tfrac{1}{2}\left[\underset{\sim}{x}'A\underset{\sim}{x} + (\underset{\sim}{x}'A\underset{\sim}{x})'\right] = \tfrac{1}{2}\left[\underset{\sim}{x}'A\underset{\sim}{x} + \underset{\sim}{x}'A'\underset{\sim}{x}\right]$$

$$= \tfrac{1}{2}\underset{\sim}{x}'(A + A')\underset{\sim}{x} = \tfrac{1}{2}\underset{\sim}{x}'(B+B')\underset{\sim}{x} = \tfrac{1}{2}(\underset{\sim}{x}'B\underset{\sim}{x} + \underset{\sim}{x}'B'\underset{\sim}{x}] = \underset{\sim}{x}'B\underset{\sim}{x}$$

Lemma 14.1.1. $\underset{n\times n}{A}$, $\underset{n\times n}{B} \equiv$ arbitrary $n\times n$ matrices, then $\underset{\sim}{x}'A\underset{\sim}{x}$, $\underset{\sim}{x}'B\underset{\sim}{x}$ in $\underset{\sim}{x}$ are identically equal iff for $j \neq i = 1,\dots,n$: $a_{ii} = b_{ii}$, $a_{ij} + a_{ji} = b_{ij} + b_{ji}$ ($A + A' = B + B'$)

$\Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'A'\underset{\sim}{x}$ $\forall x$ B symmetric $\Rightarrow A + A' = B + B' = 2B \Rightarrow B = \tfrac{1}{2}(A + A')$ | $A, B \equiv$ sym $\Rightarrow A = B$

Corollary 14.1.2. $\forall\ \underset{\sim}{x}'A\underset{\sim}{x}$ $\exists$ unique B s.t. $\underset{\sim}{x}'B\underset{\sim}{x} = \underset{\sim}{x}'A\underset{\sim}{x}$ $\forall \underset{\sim}{x}$ w/ $B = \tfrac{1}{2}(A + A')$

Corollary 14.1.3. For any pair of symmetric matrices A, B:
$\underset{\sim}{x}'A\underset{\sim}{x}$ and $\underset{\sim}{x}'B\underset{\sim}{x}$ $\equiv$ identically equal ($\underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'B\underset{\sim}{x}$ $\forall \underset{\sim}{x}$) iff $A = B$

Corollary 14.1.4. $\underset{n\times n}{A} \equiv$ Symmetric If $\underset{\sim}{x}'A\underset{\sim}{x} = 0$ $\forall \underset{\sim}{x}$, then $A = 0$
(Corollary 14.1.3. w/ $B = 0$)

Section 14.2. Nonnegative Definite Quadratic Forms and Matrices

14.2.a. Definitions

$\underset{\sim}{x}' A \underset{\sim}{x} \equiv$ quadratic form $\underset{n\times 1}{\underset{\sim}{x}} \in \mathbb{R}^n$

$\underset{\sim}{x}' A \underset{\sim}{x} \equiv$ nonnegative definite if $\underset{\sim}{x}' A \underset{\sim}{x} = 0 \;\forall\; \underset{\sim}{x} \in \mathbb{R}^n$

Note: $\underset{\sim}{x}' A \underset{\sim}{x} = 0$ for at least one $\underset{\sim}{x}$ $(\underset{\sim}{x} = \underset{\sim}{0})$

$\underset{\sim}{x}' A \underset{\sim}{x} \equiv$ non-negative definite and only takes on 0 if $\underset{\sim}{x} = \underset{\sim}{0}$ $\Rightarrow$ positive definite

$\underset{\sim}{x}' A \underset{\sim}{x} \equiv$ positive definite if $\underset{\sim}{x}' A \underset{\sim}{x} > 0 \;\forall\; \underset{\sim}{x} \neq \underset{\sim}{0}$

$\underset{\sim}{x}' A \underset{\sim}{x} \equiv$ positive semi-definite if $\underset{\sim}{x}' A \underset{\sim}{x} \geq 0 \;\forall\; x \in \mathbb{R}^n$ and $\underset{\sim}{x}' A \underset{\sim}{x} = 0$ for some $\underset{\sim}{x} \neq \underset{\sim}{0}$

Example: $\underset{\sim}{x}' I_n \underset{\sim}{x} = \sum_{i=1}^{n} x_i^2$ $\quad \underset{\sim}{x}' J \underset{\sim}{x} = (\underset{\sim}{x}' \underset{\sim}{1})(\underset{\sim}{1}' \underset{\sim}{x}) = \left(\sum_{i=1}^{n} x_i\right)^2$ both non-negative definite

$\underset{\sim}{x}' I \underset{\sim}{x} > 0 \;\forall$ non-null $\underset{\sim}{x}$ $\qquad \underset{\sim}{x}' J \underset{\sim}{x} = 0$ if $\sum_{i=1}^{n} x_i = 0$

$\underset{\sim}{x}' A \underset{\sim}{x} \equiv$ nonpositive definite, negative definite, negative semi-definite if $-\underset{\sim}{x}' A \underset{\sim}{x} \equiv$ positive ...

Quadratic form that is neither n.n.d. or n.p.d. is said to be indefinite.

$\underset{\sim}{x}' A \underset{\sim}{x} \equiv$ indefinite if $\underset{\sim}{x}' A \underset{\sim}{x} < 0$ for some $\underset{\sim}{x}$ and > 0 for some other $\underset{\sim}{x}$

(Terms applied to both quadratic forms and the matrices)

Symmetric n.n.d. matrices occur often in linear models (sums of squares) and other areas of statistics (variance-covariance matrices)

Terms apply to symmetric and nonsymmetric matrices (some presentations only apply them to symmetric)

Some presentations use n.n.d. for p.s.d.

Lemma 14.2.1. $\underset{n\times n}{D} = \{d_i\} \equiv$ diagonal matrix

(1) $D \equiv$ nonnegative definite iff $d_1, \ldots, d_n \equiv$ nonnegative
(2) $D \equiv$ positive definite iff $d_1, \ldots, d_n \equiv$ positive
(3) $D \equiv$ positive semi-definite iff $d_1, \ldots, d_n \geq 0$ w/ at least one $d_i = 0$

Suppose $\underset{n\times n}{A}$ ≡ Symmetric and is both non-negative and non-positive definite

Then $\forall \underset{n\times 1}{\underset{\sim}{x}}$ $\quad \underset{\sim}{x}'A\underset{\sim}{x} \geq 0$ and $\underset{\sim}{x}'A\underset{\sim}{x} \leq 0 \Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} = 0 \Rightarrow A = 0$

<u>Lemma 14.2.2.</u> The only $n\times n$ symmetric matrix that is both n.n.d. and n.p.d. is $\underset{n\times n}{0}$

<u>14.2.b Basic Properties of non-negative definite matrices</u>

<u>Lemma 14.2.3.</u> $k > 0$ ≡ positive scalar $\quad A$ ≡ $n\times n$ matrix

If A ≡ p.d., then kA ≡ p.d. $\quad$ If A ≡ p.s.d., then kA ≡ p.s.d.

<u>Proof</u>: $\underset{\sim}{x}'(kA)\underset{\sim}{x} = k\,\underset{\sim}{x}'A\underset{\sim}{x}$ then basic def. of p.d. and p.s.d. □

<u>Lemma 14.2.4.</u> $\underset{n\times n}{A}, \underset{n\times n}{B}$ $\quad$ If A, B ≡ n.n.d. ⇒ $A+B$ ≡ n.n.d.

If one is n.n.d. and other is p.d. ⇒ $A+B$ ≡ p.d. $\quad$ (Both p.d. too)

<u>Proof</u>: $\underset{\sim}{x}'(A+B)\underset{\sim}{x} = \underset{\sim}{x}'A\underset{\sim}{x} + \underset{\sim}{x}'B\underset{\sim}{x}$

A	B	A + B	note
p.s.d.	p.s.d.	n.n.d.	could be p.d. if different x for $=0$
p.d.	p.s.d.	p.d.	
p.s.d.	p.d.	p.d.	
p.d.	p.d.	p.d.	

□

<u>Corollary 14.2.5.</u> $A_1, \dots, A_k$ ≡ $n\times n$ matrices

If $A_1, \dots, A_k$ are all n.n.d. then $A_1 + \dots + A_k$ ≡ n.n.d. $\quad$ If atleast one is p.d. then sum is p.d.

<u>Lemma 14.2.6.</u> $\underset{n\times n}{A}\ \underset{n\times n}{B}$ w/ $B + B' = A + A'$

Then A ≡ p.d. iff B ≡ p.d. $\quad A$ ≡ p.s.d. iff B ≡ p.s.d. $\quad A$ ≡ n.n.d. iff B ≡ n.n.d.

<u>Corollary 14.2.7.</u> - Only meaningful for nonsymmetric A

A ≡ p.d. iff A' ≡ p.d. iff $\frac{1}{2}(A+A')$ ≡ p.d. $\quad$ also for p.s.d. and n.n.d.

<u>Lemma 14.2.8.</u> - Any positive definite matrix is nonsingular

<u>Proof</u>: Suppose it were not. Then $\exists\ \underset{\sim}{c} \neq \underset{\sim}{0}$ s.t. $A\underset{\sim}{c} = 0$ (linear dependencies among columns)

Then: $\underset{\sim}{c}'A\underset{\sim}{c} = 0$ for some $\underset{\sim}{c} \neq \underset{\sim}{0}$

⇒ Singular n.n.d. matrices are p.s.d. but some p.s.d. matrices are nonsingular

Theorem 14.2.9. $\underset{n\times n}{A}$, $\underset{n\times m}{P}$ 1) If $A \equiv$ n.n.d., then $P'AP \equiv$ n.n.d.

2) If $A \equiv$ n.n.d. and $\text{rank}(P) < m \Rightarrow P'AP \equiv$ p.s.d.
3) " " " $\text{rank}(P) = m \Rightarrow P'AP \equiv$ p.d.

Proof: Suppose $A \equiv$ n.n.d. $<^{\text{p.d.}}_{\text{p.s.d.}}$ $\Rightarrow \underset{\sim}{y}'A\underset{\sim}{y} \geq 0 \;\forall\; \underset{\sim}{y} \in \mathbb{R}^n$ and $\underset{\sim}{y} = \underset{n\times 1}{P\underset{\sim}{x}}$

$\Rightarrow \forall\; \underset{\sim}{x} \in \mathbb{R}^m \quad \underset{\sim}{x}'(P'AP)\underset{\sim}{x} = (P\underset{\sim}{x})'AP\underset{\sim}{x} = \underset{\sim}{y}'A\underset{\sim}{y} \geq 0 \Rightarrow P'AP \equiv$ n.n.d. 1)

If $\text{rank}(P) < m$: $\text{rank}(P'AP) \leq \text{rank}(P) < m \Rightarrow P'AP$ is not p.d. $\Rightarrow P'AP \equiv$ p.s.d. 2)

If $A \equiv$ p.d. Then $\underset{\sim}{x}'(P'AP)\underset{\sim}{x} = 0$ iff $P\underset{\sim}{x} = 0$. If $\text{rank}(P) = m$, $P\underset{\sim}{x} = \underset{\sim}{0}$ iff $\underset{\sim}{x} = \underset{\sim}{0}$ (Lem. 11.3.1)

$\Rightarrow P'AP \equiv$ p.d. (Since it is n.n.d.) $\quad \rightarrow \exists\; L$ s.t. $LP = I$
$LP\underset{\sim}{x} = L\underset{\sim}{0} \Rightarrow \underset{\sim}{x} = \underset{\sim}{0}$ $\quad \square$

Corollary 14.2.10. $\underset{n\times n}{A}$, $\underset{n\times n}{P}$ $\quad P \equiv$ nonsingular

1) If $A \equiv$ p.d. then $P'AP \equiv$ p.d. $\quad$ 2) If $A \equiv$ p.s.d. then $P'AP \equiv$ p.s.d.

Proof 1) by 14.2.9. part 3) $\quad$ 2) $A \equiv$ p.s.d. $\Rightarrow P'AP \equiv$ n.n.d. (Th. 14.2.9 1))

$\Rightarrow \exists\; \underset{\sim}{y} \neq \underset{\sim}{0}$ s.t. $\underset{\sim}{y}'A\underset{\sim}{y} = 0 \quad P$ full rank $\Rightarrow$ Let $\underset{\sim}{x} = P^{-1}\underset{\sim}{y} \Rightarrow \underset{\sim}{y} = P\underset{\sim}{x}$

for $\underset{\sim}{x} \neq 0$ since $y \neq \underset{\sim}{0} \Rightarrow \underset{\sim}{x}'(P'AP)\underset{\sim}{x} = (P\underset{\sim}{x})'AP\underset{\sim}{x} = \underset{\sim}{y}'A\underset{\sim}{y} = 0$ for some $\underset{\sim}{x} \neq 0 \Rightarrow P'AP \equiv$ p.s.d. $\quad \square$

Corollary 14.2.11. 1) A p.d. matrix is invertible and its inverse is p.d.

2) If a p.s.d. matrix is nonsingular, then its invertible, and its inverse is p.s.d.

Proof: 1) $A \equiv$ p.d. $\Rightarrow A \equiv$ nonsingular/invertible (Th. 14.2.8.)

$(A^{-1})' = (A^{-1})'AA^{-1}$ by Cor. 14.10.1. $(P = A^{-1})$

and $\text{rank}(A^{-1}) = n \Rightarrow (A^{-1})' \equiv$ p.d. $\Rightarrow A^{-1} \equiv$ p.d. (Cor 14.2.7)
2) from Cor. 14.2.10. 2) $\quad \square$

Corollary 14.2.12 - Any principal submatrix of a p.d. matrix is p.d.

Any principal submatrix of a p.s.d. matrix is n.n.d.

Proof $A \equiv n\times n$ Principal submatrix w/ rows/cols $(i_1 < i_2 < \dots < i_m)$ and $P \equiv n\times m$ matrix w/ cols $i_1, \dots, i_m$ from $I_n \equiv P'AP$ w/ $\text{rank}(P) = m$
Then from Th. 14.2.9. 1) and 3)

Corollary 14.2.13 1) Diagonal elements of p.d. matrix > 0
2) " " " p.s.d. " ≥ 0

Proof: Makes use of Cor. 14.2.12 w/ 1×1 principal submatrix w/ $\{a_{ii}\}$ $i = 1, \ldots, n$ □

Corollary 14.2.14. $\underset{n \times m}{P} \equiv$ arbitrary $P'P \equiv$ n.n.d. $\text{rank}(P) \begin{cases} = m \Rightarrow P'P \equiv \text{p.d.} \\ < m \Rightarrow P'P \equiv \text{p.s.d.} \end{cases}$

Proof: From Th. 14.2.9. w/ $P'P = P'IP$ w/ $I \equiv$ p.d.

Corollary 14.2.15. $\underset{n \times n}{P} \equiv$ nonsingular $D \equiv \{d_i\} \equiv n \times n$ diagonal matrix

1) $P'DP \equiv$ n.n.d. iff $d_1, \ldots, d_n \equiv$ nonnegative
2) $P'DP \equiv$ p.d. " " $\equiv$ positive
3) $P'DP \equiv$ p.s.d. " " $\equiv$ nonegative w/ @ least one $d_i = 0$

Proof: $A = P'DP \Rightarrow (P')^{-1} A P^{-1} = D$ Then app'ly Th. 14.2.9.1) for 1)
Cor. 14.2.10 1) and 2) for 2) and 3) □

Corollary 14.2.16. $\underset{n \times n}{A} \equiv$ symmetric $D = \{d_i\} \equiv n \times n$ diagonal. $P'AP = D$ $P \equiv$ nonsingular

1) $A \equiv$ n.n.d. iff $d_i \geq 0 \ \forall i$ 2) $A \equiv$ p.d. iff $d_i > 0 \ \forall i$ 3) $A \equiv$ p.s.d. iff $d_i \geq 0 \ \forall i$ and $d_j = 0$ for some j

Proof: From 14.2.15. w/ $A = (P')^{-1} D P^{-1}$ □

14.2.6 Nonnegative Definiteness of Symmetric Idempotent Matrices

$A \equiv$ Symmetric, idempotent matrix $\Rightarrow A = AA = A'A \Rightarrow A \equiv$ n.n.d. matrix

Lemma 14.2.17 – Every symmetric, idempotent matrix is n.n.d.

Note: I_n is the only idempotent matrix (symmetric or otherwise) that is positive definite.

Section 14.3. - Decomposition of Symmetric and Symmetric n.n.d. Matrices

From Cor. 14.2.14. : Every $\underset{n\times n}{A}$ that can be written as $P'P \equiv$ Symmetric, n.n.d.

Is every symmetric, n.n.d. A expressible as $P'P$? yes (subsection b)

Is " symmetric A expressible as $P'DP$ w/ $P \equiv$ nonsingular, $D \equiv$ diagonal? Yes.

Lemma 14.3.1. $\underset{n\times n}{A}$, $\underset{n\times n}{D} = \{d_i\} \equiv$ diagonal s.t. $A = PDQ$ for $P, Q \equiv$ nonsingular $n\times n$ matrices

Then $\text{rank}(A) = \#$ of non-zero diagonal elements in D.

Proof: Cor 8.3.3. $\underset{n\times n}{A} \equiv$ nonsingular $\underset{n\times p}{B}$: $\mathcal{R}(AB) = \mathcal{R}(B)$ and $\text{rank}(AB) = \text{rank}(B)$ (1)

$\underset{n\times n}{B} \equiv$ nonsingular, $\underset{m\times n}{A}$: $\mathcal{C}(AB) = \mathcal{C}(A)$ and $\text{rank}(AB) = \text{rank}(A)$ (2)

$\Rightarrow \text{rank}(A) = \text{rank}(PDQ) = \text{rank}(DQ) = \text{rank}(D)$ by (1) and (2) $\text{rank}(D) = \#$ of non-zero diagonal values $\square$

14.3.a. Decomposition of Symmetric Matrices

Lemma 14.3.2. $\underset{n\times n}{A} = \{a_{ij}\} \equiv$ symmetric matrix w/ $n \geq 2$

Define $B = \{b_{ij}\} = L'AL$ w/ $L \equiv n\times n$ unit lower triangular.

Let: $L = \begin{bmatrix} 1 & 0 \\ \underset{\sim}{l} & I_{n-1} \end{bmatrix}$ $A = \begin{bmatrix} a_{11} & \underset{\sim}{a}' \\ \underset{\sim}{a} & A_{22} \end{bmatrix}$ Suppose $a_{11} = 0$, $\underset{\sim}{a} \neq 0$ w/ $a_{j1} \neq 0$ for some $j \neq 1$

Suppose $j = k$. Then $\underset{(n-1)\times 1}{l}$ can be chosen so $b_{11} \neq 0$ by taking l_{k-1} to be non-zero scalar c s.t. $ca_{kk} \neq -2a_{k1}$ and by setting remaining $n-2$ elements of l to zero.

Proof: $b_{11} = [1 \ \underset{\sim}{l}'] \begin{bmatrix} a_{11} & \underset{\sim}{a}' \\ \underset{\sim}{a} & A_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \underset{\sim}{l} \end{bmatrix} = [a_{11} + \underset{\sim}{l}'\underset{\sim}{a} \quad \underset{\sim}{a}' + l'A_{22}] \begin{bmatrix} 1 \\ \underset{\sim}{l} \end{bmatrix} = a_{11} + \underset{\sim}{l}'\underset{\sim}{a} + \underset{\sim}{a}'\underset{\sim}{l} + \underset{\sim}{l}'A_{22}\underset{\sim}{l}$

$= 0 + l'\underset{\sim}{a} + \underset{\sim}{a}'l + l'A_{22}l = 0 + ca_{k1} + ca_{k1} + c^2 a_{kk} = c(2a_{k1} + ca_{kk})$

$\neq 0$ since $ca_{kk} \neq -2a_{k1}$

Lemma 14.3.3. $\underset{n\times n}{A} = \{a_{ij}\} \equiv$ Symmetric w/ $n \geq 2$ $B = U'AU$ w/ $\underset{n\times n}{U} \equiv$ upper unit triangular

Let $U = \begin{bmatrix} 1 & \underset{\sim}{u}' \\ \underset{\sim}{0} & I_{n-1} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & \underset{\sim}{b}' \\ \underset{\sim}{b} & B_{22} \end{bmatrix}$ $A = \begin{pmatrix} a_{11} & \underset{\sim}{a}' \\ \underset{\sim}{a} & A_{22} \end{pmatrix}$

suppose $a_{11} \neq 0$. Then $\underset{(n-1)\times 1}{\underset{\sim}{u}}$ can be chosen so $\underset{\sim}{b} = \underset{\sim}{0}$ by taking $\underset{\sim}{u} = -a_{11}^{-1}\underset{\sim}{a}$

Proof $B = \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{u} & I_{n-1} \end{bmatrix} \begin{bmatrix} a_{11} & \underset{\sim}{a}' \\ \underset{\sim}{a} & A_{22} \end{bmatrix} \begin{bmatrix} 1 & \underset{\sim}{u}' \\ \underset{\sim}{0} & I_{n-1} \end{bmatrix} = \begin{bmatrix} a_{11} & \underset{\sim}{a}' \\ a_{11}\underset{\sim}{u} + \underset{\sim}{a} & \underset{\sim}{u}\underset{\sim}{a}' + A_{22} \end{bmatrix} \begin{bmatrix} 1 & \underset{\sim}{u}' \\ \underset{\sim}{0} & I_{n-1} \end{bmatrix}$

$= \begin{bmatrix} a_{11} & a_{11}\underset{\sim}{u}' + \underset{\sim}{a}' \\ a_{11}\underset{\sim}{u} + \underset{\sim}{a} & a_{11}\underset{\sim}{u}\underset{\sim}{u}' + \underset{\sim}{a}\underset{\sim}{u}' + \underset{\sim}{u}\underset{\sim}{a}' + A_{22} \end{bmatrix} \Rightarrow \underset{\sim}{b} = \underset{\sim}{0}$ if $u = -\frac{1}{a_{11}}\underset{\sim}{a}$ $\triangledown$

Theorem 14.3.4. $\underset{n\times n}{A} \equiv$ Symmetric $\Rightarrow \exists$ non-singular Q s.t. $Q'AQ \equiv$ diagonal

Proof: (Induction) True for 1×1 matrix

suppose true for $(n-1)\times(n-1)$ symmetric matrix $\underset{n\times n}{A} = \{a_{ij}\} = \begin{bmatrix} a_{11} & \underset{\sim}{a}' \\ \underset{\sim}{a} & A_{22} \end{bmatrix}$

Case 1 $\underset{\sim}{a} = \underset{\sim}{0} \Rightarrow A_{22} \equiv (n-1)\times(n-1)$ symmetric by supposition $\exists$ Q^* (full rank) s.t. $Q^{*\prime}AQ^* \equiv$ diagonal

Let $Q = \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q^* \end{bmatrix} \Rightarrow Q \equiv$ nonsingular w/ $Q'AQ = \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q^{*\prime} \end{bmatrix} \begin{bmatrix} a_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & A_{22} \end{bmatrix} \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q^* \end{bmatrix} = \begin{bmatrix} a_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q^{*\prime}A_{22}Q^* \end{bmatrix}$ (diagonal)

Case 2 $\underset{\sim}{a} \neq 0, a_{11} \neq 0$ by Lem. 14.3.3. $\exists$ U s.t. $U'AU = \mathrm{diag}\begin{bmatrix} b_{11} \\ B_{22} \end{bmatrix}$

($\underset{\sim}{b} = \underset{\sim}{0}$ if $\underset{\sim}{u} = -\frac{1}{a_{11}}\underset{\sim}{a}$) by supposition $\exists$ Q_* s.t. $Q_*'B_{22}Q_* \equiv$ diagonal

Let $Q = U\mathrm{diag}(1, Q_*) \Rightarrow Q'AQ = \mathrm{diag}(1, Q_*')U'AU\mathrm{diag}(1, Q_*)$

$= \begin{bmatrix} b_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q_*'B_{22} \end{bmatrix} \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q_* \end{bmatrix} = \begin{bmatrix} b_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q_*'B_{22}Q_* \end{bmatrix} \equiv$ diagonal

Case 3 $\underset{\sim}{a} \neq 0$, but $a_{11} = 0$ Let $B = L'AL$ $L \equiv$ Lower unit triangular w/ $b_{11} \neq 0$ (Lemma 14.3.2)

By Lem 14.3.3. $\exists$ unit upper triangular U, s.t. $U'BU = \begin{bmatrix} c_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & C_{22} \end{bmatrix}$ $\rightarrow$

Theorem 14.3.4. case 3 continued By supposition $\exists\ Q_*$ s.t. $Q_*' C_{22} Q_* \equiv$ diagonal

Let $Q = LU \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q_* \end{bmatrix} \equiv$ nonsingular (product of 3 nonsingular matrices)

$$\Rightarrow Q'AQ = \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q_*' \end{bmatrix} U' \underbrace{\underline{L'AL}}_{\substack{B \\ \begin{bmatrix} C_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & C_{22} \end{bmatrix}}} U \begin{bmatrix} 1 & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q_* \end{bmatrix} = \begin{bmatrix} C_{11} & \underset{\sim}{0}' \\ \underset{\sim}{0} & Q_*' C_{22} Q_* \end{bmatrix} \equiv \text{diagonal} \quad \square$$

Note $Q \equiv$ nonsingular s.t. $Q'AQ = D \Rightarrow (Q')^{-1} Q'AQQ^{-1} = A = (Q')^{-1} D Q^{-1}$

Corollary 14.3.5. $\underset{n\times n}{A} \equiv$ symmetric $\Rightarrow \exists$ nonsingular P, diagonal D s.t. $A = P'DP$

Corollary 14.3.6. $\underset{n\times n}{A}$ then $\exists$ $P \equiv$ nonsingular and n scalars $d_1, \dots, d_n$ s.t. quadratic form $\underset{\sim}{x}'A\underset{\sim}{x}$ in $\underset{\sim}{x}$ can be written as $\underset{\sim}{x}'A\underset{\sim}{x} = \sum_{i=1}^{n} d_i y_i^2$ where $\underset{\sim}{y} = P\underset{\sim}{x}$

Proof: $\exists$ unique symmetrix B s.t. $\underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'B\underset{\sim}{x}$ $\forall \underset{\sim}{x}$ w/ $B = \frac{1}{2}(A + A')$ Cor 14.1.2.

by Cor. 14.3.5. $\exists$ nonsingular P, diagonal D s.t. $B = P'DP$

$$\Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'B\underset{\sim}{x} = \underset{\sim}{x}'P'DP\underset{\sim}{x} = \underset{\sim}{y}'D\underset{\sim}{y} = \sum_{i=1}^{n} d_i y_i^2 \quad \text{where } D = \text{diag}\{d_i\} \quad \square$$

14.3.b. Decomposition of symmetric non-negative definite matrices

Theorem 14.3.7. $\underset{n\times n}{A} \equiv$ symmetric, nonnull, n.n.d. of rank r iff

$\exists$ $\underset{r\times n}{P}$ w/ rank$(P) = r$ s.t. $A = P'P$

Proof: Suppose $A \equiv$ n.n.d. w/ rank$(A) = r \Rightarrow \exists$ nonsingular T, diagonal D s.t. $A = T'DT$ (cor. 14.3.5.)

Let $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ $T = \begin{bmatrix} \underset{\sim}{t}_1' \\ \vdots \\ \underset{\sim}{t}_n' \end{bmatrix}$ Lem. 14.3.1. $\Rightarrow$ r non-zero (positive) elements among $d_1, \dots, d_n$ say indices $k_1, \dots, k_r$

Let $P = \begin{bmatrix} \sqrt{d_{k_1}}\, \underset{\sim}{t}_{k_1}' \\ \vdots \\ \sqrt{d_{k_r}}\, \underset{\sim}{t}_{k_r}' \end{bmatrix}$. Then $A = \sum_{k=1}^{n} d_k \underset{\sim}{t}_k \underset{\sim}{t}_k' = \sum_{i=1}^{r} d_{k_i} \underset{\sim}{t}_{k_i} \underset{\sim}{t}_{k_i}' = \sum_i (\sqrt{d_{k_i}}\, \underset{\sim}{t}_{k_i})(\sqrt{d_{k_i}}\, \underset{\sim}{t}_{k_i}')$

$= P'P$ and rank$(P) = $ rank$(P'P) = $ rank$(A) = r$ $\rightarrow$

Proof of theorem 14.3.7. Continued

Conversely, if $\exists \underset{r\times n}{P}$ of rank r s.t. $A = P'P$, then:

$\text{rank}(A) = \text{rank}(P'P) = \text{rank}(P) = r$ $A \equiv$ Symmetric and by Cor. 14.2.14, $A \equiv$ n.n.d. □

Corollary 14.3.8. $\underset{n\times n}{A}$ = Symmetric, n.n.d. iff $\exists$ P w/ n columns such that $A = P'P$ ($0 = 0'0$ allows for $A \equiv$ null matrix)

Corollary 14.3.9. $\underset{n\times n}{A}$ $r = \text{rank}(A+A')$ assuming $r > 0$:

$\underset{\sim}{x}'A\underset{\sim}{x}$ in $\underset{\sim}{x}$ is n.n.d. iff for some $\underset{r\times n}{P}$ of rank r, $\underset{\sim}{x}'A\underset{\sim}{x} = (P\underset{\sim}{x})'P\underset{\sim}{x}$ $\forall \underset{\sim}{x}$

$\Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} \equiv$ sum of square elements of $P\underset{\sim}{x}$

Proof $\underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'B\underset{\sim}{x}$ $\forall \underset{\sim}{x}$ if $B = \frac{1}{2}(A+A')$ - Cor. 14.1.2.

suppose $\underset{\sim}{x}'A\underset{\sim}{x}$ or equivalently $\underset{\sim}{x}'B\underset{\sim}{x} \equiv$ n.n.d. $\Rightarrow B \equiv$ n.n.d. matrix.

$\Rightarrow$ (Th. 14.3.7.) $\exists \underset{r\times n}{P}$ of rank r s.t. $B = P'P \Rightarrow \underset{\sim}{x}'B\underset{\sim}{x} = (P\underset{\sim}{x})'P\underset{\sim}{x} = \underset{\sim}{x}'A\underset{\sim}{x}$ $\forall \underset{\sim}{x}$

Conversely Suppose for some $\underset{r\times n}{P}$ of rank r, $\underset{\sim}{x}'A\underset{\sim}{x} = (P\underset{\sim}{x})'P\underset{\sim}{x}$ $\forall \underset{\sim}{x}$ $(\underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'P'P\underset{\sim}{x})$

since $P'P \equiv$ n.n.d. (Th. 14.3.17 or Cor. 14.2.14), then $\underset{\sim}{x}'P'P\underset{\sim}{x} = \underset{\sim}{x}'A\underset{\sim}{x} \equiv$ n.n.d. quadform □

Corollary 14.3.10. $\underset{n\times n}{A}$ $r = \text{rank}(A)$ $m \geq r \equiv$ integer

If $A \equiv$ Symmetric and n.n.d., then $\exists \underset{m\times n}{P}$ s.t. $A = P'P$

Proof: Suppose $A \equiv$ Symmetric, n.n.d. and $r > 0$ ($r = 0 \Rightarrow A = 0'0$)

Th. 14.3.7 $\Rightarrow \exists \underset{r\times n}{P}$ s.t. $A = P_1'P_1$ Let $P = \begin{bmatrix} P_1 \\ 0 \end{bmatrix} \Rightarrow P'P = (P_1' \; 0')\begin{bmatrix} P_1 \\ 0 \end{bmatrix} = P_1'P_1 = A$ □

Corollary 14.3.11. $\underset{n\times m}{X}$, $\underset{n\times n}{A}$ = Symmetric, n.n.d. then $AX = 0$ iff $X'AX = 0$

Proof: Cor. 14.3.8. $\Rightarrow$ P w/ n cols exists s.t. $A = P'P \Rightarrow X'AX = X'P'PX = (PX)'PX$

Thus if $X'AX = 0 \Rightarrow AX = P'(PX) = 0$ (Cor. 5.3.2. $A = 0 \Leftrightarrow A'A = 0$). Clearly $AX = 0 \Rightarrow X'AX = 0$ □

Corollary 14.3.12. Symmetric, n.n.d. matrix is: P.d. iff nonsingular; P.s.d. iff singular

Proof: $\underset{n\times n}{A} \equiv$ symmetric, n.n.d. $A \equiv$ p.d. $\Rightarrow A \equiv$ nonsingular (Lem. 14.2.8.) (Cor. 14.3.?)

Conversely, suppose $A \equiv$ nonsingular. If $\underset{\sim}{x}'A\underset{\sim}{x} = 0$, then $A\underset{\sim}{x} = \underset{\sim}{0}$ and $\underset{\sim}{x} = A^{-1}A\underset{\sim}{x} = \underset{\sim}{0}$
$\Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} \equiv$ p.d. and $A \equiv$ positive definite $\square$

Corollary 14.3.13. $\underset{n\times n}{A} \equiv$ Symmetric, p.d. iff $\exists$ nonsingular P s.t. $A = P'P$

Proof: From Th. 14.3.7 w/ $r = n$, Cor. 14.3.12. $\square$

Corollary 14.3.14. Quadratic form $\underset{\sim}{x}'A\underset{\sim}{x}$ for $\underset{\sim}{x} \in \mathbb{R}^n$ is p.d. iff:

for some nonsingular $\underset{n\times n}{P}$, $\underset{\sim}{x}'A\underset{\sim}{x} = (P\underset{\sim}{x})'P\underset{\sim}{x} \;\forall \underset{\sim}{x}$

Proof: $\underset{\sim}{x}'A\underset{\sim}{x} \equiv$ p.d. $\Rightarrow \frac{1}{2}(A+A') \equiv$ p.d. (Cor. 14.2.7) and by Lem. 14.2.8 is of rank n

Then by Cor. 14.3.9, for some $\underset{n\times n}{P}$, $\underset{\sim}{x}'A\underset{\sim}{x} = (P\underset{\sim}{x})'P\underset{\sim}{x} \;\forall \underset{\sim}{x}$
Conversely, if for some nonsingular $\underset{n\times n}{P}$, $\underset{\sim}{x}'A\underset{\sim}{x} = (P\underset{\sim}{x})'P\underset{\sim}{x} \;\forall x$, then since
$P'P \equiv$ p.d. (Cor. 14.2.14 or Cor. 14.3.13), then $\underset{\sim}{x}'A\underset{\sim}{x} = \underset{\sim}{x}'P'P\underset{\sim}{x} \equiv$ p.d. Quadratic form $\square$

Section 14.4 – G-inverses of Symmetric non-negative definite matrices

Symmetric, p.d. matrix $\equiv$ invertible, and its inverse is p.d. and symmetric (Cor. 14.2.11, Result 8.2.4)
" p.s.d. " $\equiv$ not invertible. Are some/all of g-inverses of p.s.d. n.n.d.?

Let $\underset{n\times n}{A} \equiv \text{rank}(A) = r$ $\quad A = PDQ$ (P, Q $n\times n$ full rank matrices, $D \equiv$ diagonal) (4.1)

Lem. 14.3.1. $\Rightarrow$ r diagonal elements of $D > 0$ (say $d_{i_1}, \dots, d_{i_r}$)
Let $S = \{i_1, \dots, i_r\}$, $\bar{S} \equiv$ complement of S {integers from $1, \dots, n$ not $\in S$}
Let $D^* = \text{diag}\{d_1^*, \dots, d_n^*\}$ $\quad i \in S \Rightarrow d_i^* = 1/d_i \quad i \in \bar{S} \Rightarrow d_i^* \equiv$ arbitrary scalar

Clearly $DD^*D = D \Rightarrow D^* \equiv$ g-inverse of D
Let $G = Q^{-1}D^-P^{-1}$ and $G^* = Q^{-1}D^*P^{-1}$ $\quad AGA = PDQQ^{-1}D^-P^{-1}PDQ = PDQ = A$
$\Rightarrow G$ and $G^* \equiv$ G-inverses of A
Suppose $A \equiv$ symmetric s.t. $P = Q' \Rightarrow A = Q'DQ \Rightarrow G^* = Q^{-1}D^*(Q^{-1})'$
$\Rightarrow G^* \equiv$ Symmetric g-inverse of A $\quad d_i^* \neq 0 \;\forall i \in \bar{S} \Rightarrow G^* \equiv$ nonsingular too.

Next suppose $A \equiv$ symmetric, n.n.d. w/ $P = Q'$ $\Rightarrow A = Q'DQ$

From Cor. 14.2.15 $d_i^* > 0 \ \forall i \in S$. If $d_i^* > 0 \ \forall i \in \bar{S}$, then

$G^* \equiv$ symmetric p.d. g-inverse of A. If at least one d_i^* chosen $= 0$ for $i \in \bar{S}$,

$G^* \equiv$ " p.s.d. " " " If " " " " < 0 " "

$G^* \equiv$ indefinite

Lemma 14.4.1. Every symmetric n.n.d. matrix has a symmetric p.d. g-inverse.

Section 14.5 - LDU, U'DU, and Cholesky Decompositions

Every symmetric n.n.d. matrix A can be written as $A = P'P$

Stronger versions include: $A = T'T$ where $T \equiv$ upper triangular (This section)

$A = R^2$ where $R \equiv$ symmetric n.n.d. (Section 21.9)

LDU Decomposition: $A = LDU$ ($L \equiv$ lower unit triangular, $D = $ Diag, $U \equiv$ upper unit triangular)

U'DU " : $A = U'DU$

14.5.a. Preliminaries

Lemma 14.5.1. $\underset{n\times n}{A}$, $\underset{n\times n}{L} \equiv$ lower unit triangular $\underset{n\times n}{U} \equiv$ upper unit triangular $\underset{n\times n}{D} = \text{diag}\{d_i\}$
$(n \geq 2)$

Partition: $A = \begin{bmatrix} A_* & \underset{\sim}{a} \\ \underset{\sim}{b}' & c \end{bmatrix}$, $L = \begin{bmatrix} L_* & 0 \\ \underset{\sim}{\ell}' & 1 \end{bmatrix}$, $U = \begin{bmatrix} U_* & \underset{\sim}{u} \\ \underset{\sim}{0}' & 1 \end{bmatrix}$, $D = \begin{bmatrix} D_* & 0 \\ \underset{\sim}{0}' & k \end{bmatrix}$

w/ $A_*, L_*, U_*, D_* \equiv (n-1)\times(n-1)$ (5.1)

Then $A = LDU$ iff $L_* D_* U_* = A^*$, $L_* D_* \underset{\sim}{u} = \underset{\sim}{a}$, $U_*' D_* \underset{\sim}{\ell} = \underset{\sim}{b}$, $c - \underset{\sim}{\ell}' D_* \underset{\sim}{u} = k$

Proof: $LDU = \begin{bmatrix} L_* D_* & 0 \\ \underset{\sim}{\ell}' D_* & k \end{bmatrix}\begin{bmatrix} U_* & \underset{\sim}{u} \\ \underset{\sim}{0}' & 1 \end{bmatrix} = \begin{bmatrix} L_* D_* U_* & L_* D_* \underset{\sim}{u} \\ \underset{\sim}{\ell}' D_* U_* & \underset{\sim}{\ell}' D_* \underset{\sim}{u} + k \end{bmatrix} = \begin{bmatrix} A^* & \underset{\sim}{a} \\ \underset{\sim}{b}' & c \end{bmatrix}$

($L_* D_* U_* \equiv$ LDU decomposition of A_* as
$L_* \equiv$ lower unit triangular, $D_* \equiv$ diagonal), $U_* \equiv$ upper unit triangular)

Theorem 14.5.2. $\underset{n\times n}{A} = \begin{bmatrix} A_* & a \\ b' & c \end{bmatrix}$ $\quad A_* \equiv (n-1)\times(n-1)$

(1) If A_* has LDU decomposition, say $A_* = L_* D_* U_*$ and if $a \in \mathcal{C}(A_*)$, $b' \in \mathcal{R}(A_*)$ then $\exists$ vectors l, u s.t. $U_*' D_* l = b$ and $L_* D_* u = a$

Taking such l and u and $k = c - l' D_* u$ gives LDU decomposition of A

$$A = \begin{bmatrix} L_* & 0 \\ l' & 1 \end{bmatrix}\begin{bmatrix} D^* & 0 \\ 0' & k \end{bmatrix}\begin{bmatrix} U_* & u \\ 0' & 1 \end{bmatrix} \qquad (5.2)$$

(1') If $A_* \equiv$ symmetric ($A_*' = A_*$ and $a = b$), if A_* has $U'DU$ decomposition, say $A_* = U_*' D_* U_*$, and if $a \in \mathcal{C}(A_*)$, then there exists u s.t. $U_*' D_* u = a$ and taking such u and $k = c - u' D_* u$, a $U'DU$ decomposition of $A \equiv$

$$A = \begin{bmatrix} U_* & u \\ 0' & 1 \end{bmatrix}'\begin{bmatrix} D_* & 0 \\ 0' & k \end{bmatrix}\begin{bmatrix} U_* & u \\ 0' & 1 \end{bmatrix} \qquad (5.3)$$

(2) A has LDU decomposition iff A_* has LDU decomp, $a \in \mathcal{C}(A_*)$, $b' \in \mathcal{R}(A_*)$

(2') " " $U'DU$ " " A symmetric, A_* has $U'DU$ decomp, $a \in \mathcal{C}(A_*)$

Proof of (1): $a \in \mathcal{C}(A_*) \Rightarrow \exists\ r$ s.t. $A_* r = a$

$b' \in \mathcal{R}(A_*) \Rightarrow \exists\ s$ s.t. $b' = s' A_*$

$U_*' D_* (L_*' s) = A_*' s = b \qquad L_* D_* (U_* r) = A_* r = a$

$\Rightarrow \exists\ l$ and u s.t. $U_*' D_* l = b$ and $L_* D_* u = a$

Namely: $l = L_*' s$ and $u = U_* r$ and Lemma 14.5.1 holds.

Proof of (1'): $A_*' = A_* \Rightarrow a \in \mathcal{C}(A_*) \Rightarrow a' \in \mathcal{R}(A_*)$. Set $b = a$, $L_* = U_*'$, $l = u$

Proof of (2) and (2'): Suppose $A = LDU$ and partition L, D, and U as in (5.2)

By Lemma 14.5.1., $A_* = L_* D_* U_* \Rightarrow A_*$ has LDU decomposition

Further: $a = L_* D_* u = A_* U_*^{-1} u \in \mathcal{C}(A_*)$, $b' = l' D_* U_* = l' L_*^{-1} A_* \in \mathcal{R}(A_*)$

when $L = U'$ $(A = LDU = U'DU) \Rightarrow L_* = U_*'$, $A_* = U_*' D_* U_* \Rightarrow A_*$ has $U'DU$ decomp

and A is clearly symmetric $\square$

Theorem 14.5.3. $\underset{n\times n}{A}$ $(n \geq 2)$ $A_{11} \equiv k\times k$ leading principal submatrix of $(1 \leq k \leq n-1)$

Suppose A has an LDU decomposition, say $A = LDU$ w/:

$$L = \begin{array}{c} \\ k \\ n-k \end{array}\begin{array}{cc} k & n-k \\ \left[\begin{matrix} L_{11} & 0 \\ L_{21} & L_{22} \end{matrix}\right] \end{array}, \quad U = \begin{array}{c} \\ k \\ n-k \end{array}\begin{array}{cc} k & n-k \\ \left[\begin{matrix} U_{11} & U_{12} \\ 0 & U_{22} \end{matrix}\right] \end{array}, \quad D = \begin{array}{cc} k & n-k \\ \left[\begin{matrix} D_1 & 0 \\ 0 & D_2 \end{matrix}\right] \end{array}\begin{array}{c} \\ k \\ n-k \end{array}$$

Then an LDU decomposition of $A_{11} = L_{11} D_1 U_{11}$

Proof: $LDU = \begin{bmatrix} L_{11} D_1 & 0 \\ L_{21} D_1 & L_{22} D_2 \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} L_{11} D_1 U_{11} & L_{11} D_1 U_{12} \\ L_{21} D_1 U_{11} & L_{21} D_1 U_{12} + L_{22} D_2 U_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$\Rightarrow A_{11} = L_{11} D_1 U_{11}$ $\square$

14.5.b. Existence, Recursive construction, Uniqueness of LDU, U'DU decomposition

Theorem 14.5.4. $\underset{n\times n}{A}$ $(n \geq 2)$ $A_i \equiv$ leading principal submatrix of order i $(i = 1, \dots, n)$

For $i = 2, \dots, n$ take $\underset{\sim}{a_i} = [a_{1i}, \dots, a_{i-1,i}]'$ $\underset{\sim}{b_i}' = [a_{i1}, \dots, a_{i,i-1}]$

$\Rightarrow A_i = \begin{bmatrix} A_{i-1} & \underset{\sim}{a_i} \\ \underset{\sim}{b_i}' & a_{ii} \end{bmatrix}$

(1) Let $L_1 = (1)$, $U_1 = (1)$, $D_1 = (a_{11})$ and suppose for $i = 2, \dots, n$ $\underset{\sim}{a_i} \in \mathcal{C}(A_{i-1})$, $\underset{\sim}{b_i}' \in \mathcal{R}(A_{i-1})$

Then for $i = 2, \dots, n$ $\exists$ Unit lower triangular L_i, unit upper triangular U_i, diag D_i

s.t. $L_i = \begin{bmatrix} L_{i-1} & 0 \\ \ell_i' & 1 \end{bmatrix}$ $U_i = \begin{bmatrix} U_{i-1} & \underset{\sim}{u_i} \\ \underset{\sim}{0}' & 1 \end{bmatrix}$ $D_i = \begin{bmatrix} D_{i-1} & 0 \\ 0 & d_i \end{bmatrix}$

where $U_{i-1}' D_{i-1} \underset{\sim}{\ell_i} = \underset{\sim}{b_i}$ $L_{i-1} D_{i-1} \underset{\sim}{u_i} = \underset{\sim}{a_i}$ $d_{ii} = a_{ii} - \underset{\sim}{\ell_i}' D_{i-1} \underset{\sim}{u_i}$

and $A_i = L_i D_i U_i$ (with those matrices) $\equiv$ LDU decomposition

when $i = n \Rightarrow A = L_n D_n U_n$

(1') $U_1 = (1)$, $D_1 = (a_{11})$, $A \equiv$ Symmetric ($A_{i-1}' = A_{i-1}$ and $\underset{\sim}{b}_i = \underset{\sim}{a}_i$ $i=2,\ldots,n$)

$a_i \in \mathcal{C}(A_{i-1})$. Then for $i=2,\ldots,n$ $\exists$ U_i (unit upper triangular), diag D_i

s.t. $U_i = \begin{bmatrix} U_{i-1} & \underset{\sim}{u}_i \\ \underset{\sim}{0}' & 1 \end{bmatrix}$, $D_i = \begin{bmatrix} D_{i-1} & 0 \\ 0 & d_i \end{bmatrix}$ $U_{i-1}' D_{i-1} \underset{\sim}{u}_i = \underset{\sim}{a}_i$

$d_i = a_{ii} - \underset{\sim}{u}_i' D_{i-1} \underset{\sim}{u}_i$

$\Rightarrow A_i = U_i' D_i U_i$, $A = U_n' D_n U_n$

(2) A has LDU decomposition only if, for $i=2,\ldots,n$, $a_i \in \mathcal{C}(A_{i-1})$, $\underset{\sim}{b}_i' \in \mathcal{R}(A_{i-1})$

(2') " " $U'DU$ " " ", " " $\underset{\sim}{a}_i \in \mathcal{C}(A_{i-1})$

Example: Let $A \equiv 2\times 2$ w/ $a_{11} = 0 \Rightarrow$ A has LDU decomposition iff $a_{12} = a_{21} = 0$

$\Rightarrow A = \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix}$ $U = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$

w/ arbitrary ℓ, u $1(0)(1) = 0 = A_{11}$ $1(0)u = 0 = a_{12}$ $1(0)\ell = 0 = a_{21}$

$a_{22} - \ell(0)u = K$ (d_{22})

__Theorem 14.5.5.__ $\underset{n\times n}{A} \equiv$ matrix of rank r w/ an LDU decomposition

$\underset{n\times n}{L} \equiv$ unit lower triangular, $\underset{n\times n}{U} \equiv$ unit upper triangular, $\underset{n\times n}{D} \equiv$ diagonal

s.t. $A = LDU$ 1) $D \equiv$ unique

2) Suppose the $i_1 < \cdots < i_r$ elements of D are nonzero and others are 0.

Then for $i \in \{i_1,\ldots,i_r\}$ and $j > i$, u_{ij} and $l_{ji} \equiv$ unique

and for $i \notin \{i_1,\ldots,i_r\}$ ——— $\equiv$ arbitrary

$\Rightarrow$ Elements above diagonal in U in rows $\{i_1,\ldots,i_r\} \equiv$ unique others arbitrary

" below " " L " Columns $\{i_1,\ldots,i_r\} \equiv$ " " "

__Corollary 14.5.6.__ If $\underset{n\times n}{A}$ has an LDU decomposition and if leading principal submatrix of A ~~of~~ order $n-1$ is singular, then LDU decomposition is unique.

Proof: $D = \begin{bmatrix} D_{11} & 0 \\ 0 & d \end{bmatrix}$ and ~~[illegible]~~ all diagonal elements of $D_{11} \neq 0$ $\square$

Corollary 14.5.7. $\underset{n\times n}{A}$ w/ $n \geq 2$ $A_i \equiv$ leading principal submatrix of A $i=1,\dots,n-1$

If $A_1, A_2, \dots, A_{n-1} \equiv$ nonsingular, then A has unique LDU decomposition.

Lemma 14.5.8. $A \equiv$ Symmetric w/ an unique LDU decomposition $A = LDU$, then:

$L = U'$ and $A = U'DU$ has a $U'DU$ decomposition.

Proof: $A = LDU$ A symmetric $\Rightarrow A = A' = U'D'L' \equiv LDU$ decomp. ($U' =$ lower, $L' \equiv$ upper)

Uniqueness $\Rightarrow L = U' \equiv U = L'$ □

Notes: $A = LDU = L^*U = LU^*$ ($L^* = LD \equiv$ Lower triangular) not necessarily unit
($U^* = DU \equiv$ upper ")

LU decomposition $A = LU$ (L, U lower, upper triangular, respectively)

$L \equiv$ lower unit triangular $\equiv$ Doolittle Decomposition
$U \equiv$ upper " " $\equiv$ Crout Decomposition

14.5.c. Decomposition of Positive Definite Matrices

Theorem 14.5.9 $\underset{n\times n}{A}$ $\equiv$ positive definite has unique LDU decomp. $A = LDU$ w/ diagonal element of D are all positive

(Lem. 14.2.8.)

Proof: Cor. 14.2.12: Any principal submatrix of p.d. matrix is p.d. and nonsingular

$\Rightarrow A$ has unique LDU decomposition: $A = LDU$ (Cor. 14.5.7)

Need to show $d_i > 0, i=1,\dots,n$. Let $B = DU(L^{-1})'$ where $(L^{-1})' \equiv$ upper unit triangular

$\Rightarrow$ diagonal elements of $B \equiv$ diagonal elements of D (diag of $U(L^{-1})' \equiv 1$s)

Now: $B = L^{-1}(LDU)(L^{-1})' = L^{-1}A(L^{-1})' \Rightarrow$ (Cor 14.2.10) that $B \equiv$ P.d.

$\Rightarrow$ (Cor. 14.2.13.) that $d_1, \dots, d_n > 0$. □

Corollary 14.5.10. $\underset{n\times n}{A}$ $\equiv$ symmetric, P.d. has a unique $U'DU$ decomposition:

$A = U'DU$ and the diagonal elements of $D \equiv$ positive.

Theorem 14.5.11. $\underset{n\times n}{A} \equiv$ symmetric, p.d. $\Rightarrow \exists$ unique upper triangular T s.t. $A = T'T$ where T has positive diagonal elements

Moreover, taking $U \equiv$ unique unit upper triangular and $D \equiv$ unique diagonal matrix s.t. $A = U'DU \Rightarrow T = D^{1/2}U$, where $D^{1/2} = \text{diag}\{\sqrt{d_i}\}$

Proof of Existence: $D^{1/2}U \equiv$ upper triangular w/ diagonal elements $\{\sqrt{d_i}\}$ and $A = (D^{1/2}U)'D^{1/2}U \Rightarrow \exists$ upper triangular T w/ positive elements and $A = T'T$

Proof of Uniqueness: suppose $\underset{n\times n}{T} \equiv$ upper triangular w/ positive diag elements and $A = T'T$

Let $D_* = \text{diag}\{t_{ii}^2\}$ and $U_* = [\text{diag}\{t_{ii}\}]^{-1}T = \text{diag}\{\frac{1}{t_{ii}}\}T$

$\Rightarrow U_* \equiv$ unit upper triangular and $A = U_*'D_*U_* \equiv U'DU$ decomp of A

Then by Cor. 14.5.10.: $D_* = D$ or equiv. $t_{ii} = \sqrt{d_i}$ $i = 1, \dots, n$ $(t_{ii} > 0)$

and that $U_* = U$. $\Rightarrow T = D_*^{1/2}U_* = D^{1/2}U \Rightarrow$ unique $T \equiv$ upper triangular w/ positive diagonal units. $\square$

- $A = T'T$ w/ $T \equiv$ upper triangular w/ positive diagonal elements $\equiv$ Cholesky Decomposition

14.5.d. Decomposition of Symmetric non-negative definite matrices

Theorem 14.5.12. – $\underset{n\times n}{A} \equiv$ Symmetric n.n.d. $\Rightarrow \exists$ unit upper triangular U s.t. $U'AU \equiv$ diagonal

Lemma 14.5.13 – $\underset{n\times n}{A} \equiv$ n.n.d. matrix. If $a_{ii} = 0$, then for $j = 1, \dots, n$ $a_{ij} = -a_{ji}$

($a_{ii} = 0 \Rightarrow$ row i: $(a_{i1}, \dots, a_{in}) = -(a_{1i}, \dots, a_{ni})$: column i

Proof of Lemma 14.5.13.: – Suppose $a_{ii} = 0$, Let $\underset{n\times 1}{\underset{\sim}{x}}$ s.t. $x_i < -a_{jj}$, $x_j = a_{ij} + a_{ji}$ $x_k = 0$ $k \neq i, j$ $(i \neq j)$

$\underset{\sim}{x}'A\underset{\sim}{x} = a_{ii}x_i^2 + (a_{ij} + a_{ji})x_ix_j + a_{jj}x_j^2 = (a_{ij} + a_{ji})^2(x_i + a_{jj}) \leq 0$

w/ equality only if $a_{ij} + a_{ji} = 0 \Rightarrow a_{ij} = -a_{ji}$ $A \equiv$ n.n.d $\Rightarrow \underset{\sim}{x}'A\underset{\sim}{x} = 0$ $\square$

Corollary 14.15.14. $\underset{n\times n}{A}$ ≡ symmetric, n.n.d. If $a_{ii}=0$, then $a_{ij}=a_{ji}=0$ $j=1,\dots,n$

Proof of Theorem 14.15.12. : Induction, similar to Theorem 14.3.4.

Corollary 14.15.15. $\underset{n\times n}{A}$ ≡ Symmetric, n.n.d. has $U'DU$ decomp: $A=U'DU$

and for any such decomposition, diagonal elements of D ≡ nonnegative.

Proof: Cor. 14.2.15. ⇒ $P'DP$ ≡ nonnegative definite ⇔ $d_1,\dots,d_n$ ≡ nonnegative

Need to show A has $U'DU$ decomposition. Theorem 14.15.12. ⇒ ∃ T ≡ unit upper triangular

s.t. $T'AT=D$ let $U=T^{-1}$ ⇒ $(T^{-1})'T'ATT^{-1}=A=U'DU$ (U ≡ unit upper triangular) □

Theorem 14.5.16. $\underset{n\times n}{A}$ = Symmetric, n.n.d., $r=\operatorname{rank}(A)$

Then ∃ unique unit upper triangular T w/ r positive diagonal elements and w/ $n-r$ null rows s.t. $A=T'T$. Taking U ≡ a unit upper triangular and D ≡ unique diagonal matrix s.t. $A=U'DU$, then $T=D^{1/2}U$.

14.5.e. Recursive Formulas for LDU and Cholesky Decompositions

Finding U, L, D s.t. $A=LDU$ when they exist in n steps.

$$d_1=a_{11} \qquad d_1u_{1j}=a_{1j} \qquad \cancel{\Rightarrow u_{1j}=\frac{a_{1j}}{d_1}\ (a_{11}\neq 0)}$$

$$d_i u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}d_k u_{kj} \quad (i=2,\dots,j-1)$$

$$d_1 l_{j1} = a_{j1}$$

$$d_i l_{ji} = a_{ji} - \sum_{k=1}^{i-1} u_{ki}d_k l_{jk} \quad (i=2,3,\dots,j-1)$$

$$d_j = a_{jj} - \sum_{k=1}^{j-1} u_{kj}d_k l_{jk} \quad (j=2,\dots,n) \qquad (5.15)$$

$d_i=0 \Rightarrow u_{ij}, l_{ji}$ ≡ arbitrary (can be set to 0).

$$d_i\neq 0 \Rightarrow u_{ij}=\left[a_{ij}-\sum_{k=1}^{i-1} l_{ik}d_k u_{kj}\right]/d_i$$

$$l_{ji}=\left[a_{ji}-\sum_{k=1}^{i-1} u_{ki}d_k l_{jk}\right]/d_i$$

Cholesky decomposition of $\underset{n\times n}{A}$ ≡ symmetric, n.n.d. ~~Find T w/ T~~ — $\mathrm{rank}(A) = r$

⇒ Find T (upper triangular w/ r positive diagonal elements, n−r null rows) s.t. $A = T'T$.

SQUARE ROOT METHOD

$$t_{11} = \sqrt{a_{11}} \qquad t_{1j} = \begin{cases} a_{1j}/t_{11} & \text{if } t_{11} > 0 \\ 0 & \text{if } t_{11} = 0 \end{cases}$$

$$t_{ij} = \begin{cases} \left(a_{ij} - \sum_{k=1}^{i-1} t_{ki}t_{kj}\right)/t_{ii} & \text{if } t_{ii} > 0 \\ 0 & \text{if } t_{ii} = 0 \end{cases} \qquad i = 1, \dots, j-1$$

$$t_{jj} = \left(a_{jj} - \sum_{k=1}^{j-1} t_{kj}^2\right)^{1/2} \qquad j = 2, \dots, n$$

Can be row-by-row or column-by-column.

Checks for non-negative definiteness of A during √ method:

1) If $a_{11} = 0$, test whether $a_{1j} = 0$ $j = 2, \dots, n$ (before computing t_{11})
2) for $2 \le i \le n$, test (before computing t_{ii}): $a_{ii} - \sum_{k=1}^{i-1} t_{ki}^2 \ge 0$
3) If $a_{ii} - \sum_{k=1}^{i-1} t_{ki}^2 = 0$, test whether $a_{ij} - \sum_{k=1}^{i-1} t_{ki}t_{kj} = 0$ (before determining t_{ij})
4) Test whether $a_{nn} - \sum_{k=1}^{n-1} t_{kn}^2 \ge 0$ (before determining t_{nn})

Example: $A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 10 \end{bmatrix}$ (Row-by-row modified square root version)

1) $a_{11} > 0 \Rightarrow t_{11} = \sqrt{4} = 2$, $t_{12} = \frac{2}{2} = 1$, $t_{13} = \frac{-2}{2} = -1$

2) $a_{22} - t_{12}^2 = 1 - (1)^2 = 0 \Rightarrow t_{22} = \left(1 - (+1)^2\right)^{1/2} = 0$

3) $a_{23} - t_{12}t_{13} = -1 - (1)(-1) = 0 \Rightarrow t_{23} = 0$

4) $a_{33} - \sum_{k=1}^{2} t_{k3}^2 = 10 - \left((-1)^2 + 0^2\right) = 9 \ge 0 \Rightarrow t_{33} = \sqrt{9} = 3$

→

$$\Rightarrow T = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad T'T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & 1 & 10 \end{bmatrix} = A$$

14.5.f. Using LDU, U'DU, or Cholesky Decomposition for G-inverses

$\underset{n \times n}{A} \equiv$ matrix w/ LDU decomposition: $A = LDU$

Let $G = U^{-1}D^{-}L^{-1} \equiv$ G-inverse of A where D^{-} replaces non-zero diagonal elements of D with $1/d_i$.

Recursive formulas above can be used to invert U and L

when $L \equiv U'$, $G = U^{-1}D^{-}(U^{-1})'$

Cholesky Decomposition: Let $\underset{r \times n}{T_1} \equiv$ matrix striking out $n-r$ null rows of T

Then $A = T_1'T_1$ w/ $\text{rank}(T_1) = \text{rank}(A) = r \Rightarrow T_1$ has right-inverse R

s.t. $T_1 R = I \Rightarrow ARR'A = T_1'\underbrace{T_1 R}_{I}\underbrace{R'T_1'}_{I}T_1 = T_1'T_1 = A$

$\Rightarrow RR' \equiv$ G-inverse of A