STA 6208 – Spring 2004 – Exam 1

Print Name: **UFID:**

All questions are based on the following two regression models, where SIMPLE REGRESSION refers to the case where $p = 1$, and X is of full column rank (no linear dependencies among the predictor variables)

$$\textbf{Model 1:} \quad Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + \varepsilon_i \quad i = 1, \ldots, n \quad \varepsilon_i \sim NID(0, \sigma^2)$$

$$\textbf{Model 2:} \quad \mathbf{Y} = \mathbf{X}\beta + \varepsilon \quad \mathbf{X} \equiv n \times p' \quad \beta \equiv p' \times 1 \quad \varepsilon \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$$

Cochran's Theorem
Suppose $\mathbf{Y}$ is distributed as follows with nonsingular matrix $\mathbf{V}$:

$$\mathbf{Y} \sim N(\mu, \mathbf{V}\sigma^2) \qquad r(\mathrm{V}) = n$$

then:

1. $\mathbf{Y}' \left(\frac{1}{\sigma^2}\mathbf{A}\right) \mathbf{Y}$ is distributed noncentral χ^2 with:
   (a) Degrees of freedom $= r(\mathbf{A})$
   (b) Noncentrality parameter $= \Omega = \frac{1}{2\sigma^2} \boldsymbol{\mu}' \mathbf{A} \, \boldsymbol{\mu}$ if $\mathbf{AV}$ is idempotent
2. $\mathbf{Y}'\mathbf{AY}$ and $\mathbf{Y}'\mathbf{BY}$ are independent if $\mathbf{AVB} = \mathbf{0}$
3. $\mathbf{Y}'\mathbf{AY}$ and linear function $\mathbf{BY}$ are independent if $\mathbf{BVA} = \mathbf{0}$

1) Based on **Model 1**, **derive** the normal equations for the simple linear regression model.

2) Show that $\sum_{i=1}^{n}(\hat{Y}_i - \overline{Y}) = 0$. You may do this based on either **Model 1** or **Model 2**.

3) A simple linear regression is fit, relating first weekend revenues (Y) to advertising expenditures (X) for $n = 5$ randomly selected horror films:

Film	i	Sales	Ad Exp
Scarier Movie	1	25.0	8.0
I Know What You Did Last Winter	2	15.0	6.0
Rural Legend	3	12.0	4.0
Shout	4	30.0	10.0
Friday the 14th	5	18.0	7.0

Give the following matrices: $\mathbf{Y}$, $\mathbf{X}$, $\mathbf{X}'\mathbf{X}$, $\mathbf{X}'\mathbf{Y}$.

4) An engineer is interested in the relationship between steel thickness (X) and its breaking strengh (Y). She obtains the following matrices from a matrix computer package:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 12 & 60 \\ 60 & 360 \end{bmatrix} \quad \mathbf{X}'\mathbf{Y} = \begin{bmatrix} 120 \\ 800 \end{bmatrix} \quad \mathbf{Y}'(\mathbf{I} - \mathbf{P})\mathbf{Y} = 20 \quad \mathbf{Y}'(\mathbf{P} - \frac{\mathbf{1}}{\mathbf{n}}\mathbf{J})\mathbf{Y} = 250$$

a) Give $\hat{\beta}$ and $s^2\{\hat{\beta}\}$

b) Give a 95% confidence interval for β_1.

c) Test $H_0 : \beta_1 = 0$ vs $H_A : \beta_1 \neq 0$ at the $\alpha = 0.05$ significance level.

5) Write out $\frac{SS(Model)}{\sigma^2}$ and $\frac{SS(Residual)}{\sigma^2}$ for **Model 2**. Use Cochran's theorem to obtain the sampling distribution for each quantity (specifically defining all relevant terms), and show that the two quantities are independent.

6) Write $\mathbf{e}$ for **Model 2** as a linear function of $\mathbf{Y}$, and use that to **derive** the mean vector and covariance matrix of $\mathbf{e}$. Are the residuals uncorrelated?