STA 6208 – Spring 2002 – Exam 2

Print Name: **SSN:**

Cochran's Theorem
Suppose $\mathbf{Y}$ is distributed as follows with nonsingular matrix $\mathbf{V}$:

$$\mathbf{Y} \sim N(\mu, \mathbf{V}\sigma^2) \qquad r(\mathrm{V}) = n$$

then:

1. $\mathbf{Y}' \left(\frac{1}{\sigma^2}\mathbf{A}\right) \mathbf{Y}$ is distributed noncentral χ^2 with:
   (a) Degrees of freedom $= r(\mathbf{A})$
   (b) Noncentrality parameter $= \Omega = \frac{1}{\sigma^2}\ \boldsymbol{\mu}'\mathbf{A}\ \boldsymbol{\mu}$ if $\mathbf{AV}$ is idempotent
2. $\mathbf{Y}'\mathbf{AY}$ and $\mathbf{Y}'\mathbf{BY}$ are independent if $\mathbf{AVB} = \mathbf{0}$
3. $\mathbf{Y}'\mathbf{AY}$ and linear function $\mathbf{BY}$ are independent if $\mathbf{BVA} = \mathbf{0}$

1) For the linear regression model:

$$\mathbf{Y} = \mathbf{X}\ \boldsymbol{\beta} +\ \boldsymbol{\varepsilon} \qquad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \boldsymbol{\sigma^2}\mathbf{I})$$

the response vector $\mathbf{Y}$ can be written as $\mathbf{Y} = \hat{\mathbf{Y}} + \mathbf{e}$.

a) Write the total uncorrected sum of squares in terms of the model and residual sums of squares. Show that the predicted vector $\hat{\mathbf{Y}}$ is orthogonal to the residual vector $\mathbf{e}$.

b) Give the distribution of $SS(\mathrm{R}egression)/\sigma^2)$.

c) Give the distribution of $SS(\mathrm{R}esidual)/\sigma^2)$.

d) Show that these two quantities are independent.

e) We wish to test $H_0:\ \boldsymbol{\beta}* = \mathbf{0}$ vs $H_A:\ \boldsymbol{\beta}* \neq \mathbf{0}$, where $\boldsymbol{\beta}*$ is the subvector of $\boldsymbol{\beta}$ containing all regression coefficients except β_0. Give the test statistic, as well as its distribution under H_0 and under H_A.

2) A study was conducted to observe the effect of Orlistat for the treatment of obesity. The following table gives the means, within group sums of squares, and sample sizes for kg lost in 12-week treatment period. X represents the dose

Statistic	$X = 30$ $(j = 1)$	$X = 180$ $(j = 2)$	$X = 360$ $(j = 3)$
$\overline{Y}_j$	3.61	3.69	4.74
$\sum(Y - \overline{Y}_j)^2$	6.79	6.69	6.64
n_j	48	45	47

$$\hat{Y} = 3.3536 + 0.0035X$$

a) Give the fitted values for each group based on the simple linear regression model.

b) Give the Pure Error sums of squares and degrees of freedom.

c) Give the Lack of Fit sums of squares and degrees of freedom.

d) Test whether the simple linear regression model gives an adequate fit to this data ($\alpha = 0.05$).

3) A company has developed a new product and wishes to measure the effects of two types of advertising: radio and print. They select a sample of $n = 8$ test markets of relatively equal sales potential and assign each a budget for radio (X_1, in \$1000s) and print (X_2, in \$1000s). The response measured is units sold (in 1000s). The data are given in the following table. The model fit is:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon \qquad \varepsilon \sim NID(0, \sigma^2)$$

i	Y_i	X_{i1}	X_{i2}
1	2.5	1.0	1.0
2	3.0	2.0	1.0
3	3.5	1.0	2.0
4	4.0	2.0	2.0
5	2.0	1.0	1.0
6	3.0	2.0	1.0
7	4.5	1.0	2.0
8	5.0	2.0	2.0

$$\mathbf{X'X} = \begin{bmatrix} 8 & 12 & 12 \\ 12 & 20 & 18 \\ 12 & 18 & 20 \end{bmatrix} \quad (\mathbf{X'X})^{-1} = \begin{bmatrix} 2.375 & -0.75 & -0.75 \\ -0.75 & 0.5 & 0.0 \\ -0.75 & 0.0 & 0.5 \end{bmatrix} \quad \hat{\boldsymbol{\beta}} = \begin{bmatrix} 0.0625 \\ 0.625 \\ 1.625 \end{bmatrix} \quad \text{SS}(Residual) = 1.15625$$

a) Set up the $\mathbf{K'}$ matrix to test whether the partial effect of radio expenditures is equal to that for newspaper expenditures.

b) Test whether the partial effect of radio expenditures is equal to that for newspaper expenditures ($\alpha = 0.05$).

c) Give simultaneous 95% confidence intervals for β_0, β_1, and β_2.

d) Give a 95% prediction interval for sales in a similar market with both radio and newspaper expenditures set at 1.5 (\$1000s).