1-Way Random Effects Models

Q.1. A 1-Way random effects model was fit, comparing copper concentrations among bottles of a Scotch whiskey brand for a given age. A random sample of 5 bottles was selected, and within each bottle 3 samples were taken, and copper concentration was measured. The following table gives the sample means and variances for the 5 bottles for the model.

$$y_{ij} = \mu + a_i + e_{ij} \quad a_i \sim NID\left(0, \sigma_a^2\right) \quad e_{ij} \sim NID\left(0, \sigma_e^2\right) \quad \{a_i\} \perp \{e_{ij}\}$$

Bottle	#reps	Mean	Variance
1	3	265	500
2	3	335	1300
3	3	270	200
4	3	355	1225
5	3	375	625

p.1.a. Compute the Among Bottle sum of squares, give its degrees of freedom, and Expected Mean Square

SSA = ________________________________ df_A = ________________________ E{MSA} = ___________________

p.1.b. Compute the Within Bottle sum of squares, give its degrees of freedom, and Expected Mean Square

SSW = ________________________________ df_W = ________________________ E{MSW} = ___________________

p.1.c. Test whether the variance of the bottle effects is 0. $H_0: \sigma_a^2 = 0$ $H_A: \sigma_a^2 > 0$

Test Statistic: ______________________________________ Rejection Region: ______________________________

p.1.d. Obtain Point estimates for: $\sigma_e^2, \quad \sigma_a^2, \quad \rho_I = \dfrac{\sigma_a^2}{\sigma_a^2 + \sigma_e^2}$

$\hat{\sigma}_e^2 =$ ___________________________ $\hat{\sigma}_a^2 =$ ________________________ $\hat{\rho}_I =$ ______________________

Q.2. Consider the 1-Way Random Effects Model:

$$y_{ij} = \mu + a_i + e_{ij} \quad i = 1,...,g \quad j = 1,...,n \quad a_i \sim NID\left(0, \sigma_a^2\right) \quad e_{ij} \sim NID\left(0, \sigma_e^2\right) \quad \{a\} \perp \{e\}$$

p.2.a. **Based on rules involving linear functions of RVs, derive** the following values:

$$E\left(y_{ij}\right), \quad V\left(y_{ij}\right), \quad V\left(y_{i\bullet}\right), \quad V\left(y_{\bullet\bullet}\right), \quad COV\left(y_{ij}, y_{i\bullet}\right)$$

p.2.b. In a population of individuals, 95% have mean values between 80 and 120. ρ_I, the Intraclass Correlation is 0.80. Within individuals, 95% of individual observations lie within how many units from the individual mean?

Q.3. For the 1-Way Random Effects model, derive the Expected Mean Squares for Treatments and Error by completing the following steps. $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad \alpha_i \sim NID\left(0, \sigma_\alpha^2\right) \quad \varepsilon_{ij} \sim NID\left(0, \sigma^2\right) \quad \{\alpha\} \perp \{\varepsilon\} \quad i = 1, \ldots, g;\ j = 1, \ldots, n$

p.3.a. Derive: $E\{Y_{ij}\}, \quad V\{Y_{ij}\}, \quad E\{Y_{i\bullet}\}, \quad V\{Y_{i\bullet}\}, \quad E\{Y_{\bullet\bullet}\}, \quad V\{Y_{\bullet\bullet}\}$ SHOW ALL WORK.

p.3.b. Making use of p.3.a., derive: $E\left\{\sum_{i=1}^{g}\sum_{j=1}^{n} Y_{ij}^2\right\}, \quad E\left\{n\sum_{i=1}^{g} \overline{Y}_{i\bullet}^2\right\}, \quad E\left\{ng\overline{Y}_{\bullet\bullet}^2\right\}$

p.3.c. Making use of p.3.b., derive: $E\{MS_{\text{TRT}}\}$ and $E\{MS_{\text{ERR}}\}$

Q.4. A 1-Way Random Effects model is fit, comparing readability scores among a random sample of g=6 Business/Economics columnists. A sample of n=3 essays were selected from each columnist and their readability was assessed based on the Flesch-Kincaid scale.

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad \alpha_i \sim N\left(0, \sigma_\alpha^2\right) \quad \varepsilon_{ij} \sim NID\left(0, \sigma^2\right) \quad \{\alpha\} \perp \{\varepsilon\} \quad \sum_{i=1}^{6}\sum_{j=1}^{3}\left(Y_{ij} - \overline{Y}_{i\bullet}\right)^2 = 35.4 \quad \sum_{i=1}^{6}\sum_{j=1}^{3}\left(Y_{ij} - \overline{Y}_{\bullet\bullet}\right)^2 = 90.1$$

p.4.a. Test $H_0: \sigma_\alpha^2 = 0 \quad H_A: \sigma_\alpha^2 > 0$

Test Statistic: ______________________________ Rejection Region: ________________________________

p.4.b. Obtain a point estimate and an approximate 95% Confidence Interval for σ_α^2 (based on Satterthwaite's Approx.)

Point Estimate: _________________ Approximate 95% Confidence Interval: _____________________________

Q.5. In a population of stock market analysts, the mean rate of return is 2%. Approximately 95% of all analysts have a personal mean return within 5% of the overall mean. Among individual stocks selected by a particular analyst, approximately 95% have returns within 7% of his/her mean.

Compute: $\sigma_\alpha^2 =$ _____________ $\sigma^2 =$ ______________ $V\{Y\} =$ _______________ $\rho_I =$ ____________

Q.6. Jack runs a small firm with 3 employees. He plans to conduct an experiment to compare them with respect to the quantity of their output using a new drill press, observing each employee's output over 5 sessions. Jill has a large firm with many employees. She plans on sampling 3 at random, and observing each employee's output over 5 sessions.

p.6.a. Clearly state Jack's (treatment effects) model, describing all elements and assumptions.

p.6.b. **Derive (Showing all work)** $E\left\{\overline{Y}_{i\bullet}\right\}, \quad V\left\{\overline{Y}_{i\bullet}\right\}, \quad E\left\{\overline{Y}_{\bullet\bullet}\right\}, \quad V\left\{\overline{Y}_{\bullet\bullet}\right\},$ for Jack's model.

p.6.c. Clearly state Jill's model, describing all elements and assumptions.

p.6.d. **Derive (Showing all work)** $E\{\overline{Y}_{i\bullet}\}, \quad V\{\overline{Y}_{i\bullet}\}, \quad E\{\overline{Y}_{\bullet\bullet}\}, \quad V\{\overline{Y}_{\bullet\bullet}\},$ for Jill's model.

Q7. A wine company produces bottles of wine that have mean alcohol contents (percentages) that are normally distributed, with 95% of all bottle means between 10.5 and 13.5. For specimens within bottles, alcohol contents are normally distributed with 95% of specimens falling within 0.6 of the bottle mean. Based on the 1-way random effects model, give:

$\sigma^2 a$ =

σ^2 =

ρ_I =

Q.8. An experiment is conducted to study variation in breaking strength of denim jeans in a large automated manufacturing facility. There are 1000s of machines. Each machine has a true mean breaking strength of jeans that it produces, and these means are normally distributed. Approximately 95% of the machines have means between 40 and 60. Within a given machine, approximately 95% of individual pairs of jeans lie within 6 of that machine's mean. The model fit is: $y_{ij} = \mu_i + e_{ij} \qquad \mu_i \sim NID\left(\mu_{\bullet}, \sigma_a^2\right) \quad e_{ij} \sim NID\left(0, \sigma^2\right) \quad \{\mu_i\} \perp \{e_{ij}\}$

For this scenario, obtain the following parameters and measures:

$\mu_{\bullet} =$ ______________ $\sigma_a^2 =$ ______________ $\sigma^2 =$ ______________ $\rho_I = Corr\left\{Y_{ij}, Y_{ij'}\right\} =$ ______________ $j \neq j'$

Q.9. Among a population of lakes, the mean adult fish lengths are normally distributed, with approximately 95% of the lake means lying between 50 and 70 centimeters. Within lakes, approximately 95 % of the fish have lengths within 12 cm of the lake mean. Consider a 1-Way random effects model, where a sample of *g* lakes is selected and *n* fish are sampled from each lake.

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad i = 1, \ldots, g; \; j = 1, \ldots, n \quad \alpha_i \sim NID\left(0, \sigma_\alpha^2\right) \quad \varepsilon_{ij} \sim NID\left(0, \sigma^2\right) \quad \{\alpha_i\} \perp \{\varepsilon_{ij}\}$$

p.9.a. Obtain $\mu, \quad \sigma_\alpha^2, \quad \sigma^2$

p.9.b. Give the mean and variance of $\overline{Y}_{\bullet\bullet}$

Q.10. Consider the 1-Way Random Effects Model:

$$y_{ij} = \mu + a_i + e_{ij} \quad i = 1,...,g \quad j = 1,...,n \quad a_i \sim NID\left(0, \sigma_a^2\right) \quad e_{ij} \sim NID\left(0, \sigma_e^2\right) \quad \{a\} \perp \{e\}$$

p.10.a. **Based on rules involving linear functions of RVs, derive** the following values:

$$E\left(y_{ij}\right), \quad V\left(y_{ij}\right), \quad V\left(y_{i\bullet}\right), \quad V\left(y_{\bullet\bullet}\right), \quad COV\left(y_{ij}, y_{i\bullet}\right)$$

p.10.b. In a population of individuals, 95% have mean values between 40 and 60. Within individuals, 95% of individual observations lie within 4 units from the individual mean. Compute ρ_I, the Intraclass Correlation