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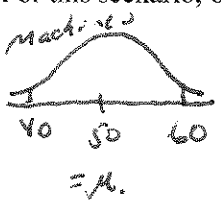
STA 6208 – Exam 2 – Spring 2015 – PRINT Name ANSWER KEY

Conduct Individual Test/CI's at α = 0.05 significance level, and all simultaneous tests/CI's @ experimentwise rate 0.05

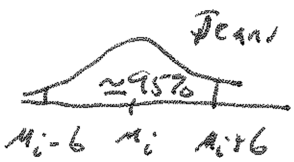
Q.1. An experiment is conducted to study variation in breaking strength of denim jeans in a large automated manufacturing facility. There are 1000s of machines. Each machine has a true mean breaking strength of jeans that it produces, and these means are normally distributed. Approximately 95% of the machines have means between 40 and 60. Within a given machine, approximately 95% of individual pairs of jeans lie within 6 of that machine's mean. The model fit is: $y_{ij} = \mu_i + e_{ij} \qquad \mu_i \sim NID(\mu_\bullet, \sigma_a^2) \quad e_{ij} \sim NID(0, \sigma^2) \quad \{\mu_i\} \perp \{e_{ij}\}$

For this scenario, obtain the following parameters and measures:



$\sigma_a \approx 5$

$\sigma_a^2 \approx 25$



$\sigma \approx 3$

$\sigma^2 \approx 9$

$\mu_\bullet =$ 50 (8) $\qquad \sigma_a^2 =$ 25 (6) $\qquad \sigma^2 =$ 9 (6) $\qquad \rho_I = Corr\{Y_{ij}, Y_{ij'}\} = \frac{25}{25+9} = 0.735 \quad j \neq j'$ (6)

Q.2. An experiment is conducted to estimate the effects of 3 brands of shoes on one mile run times for a sample of 4 runners. Each runner's time is measured 5 times in each brand of shoe. The experimenters treat the shoes as fixed and the runners as random.

p.2.a. Write out the statistical model, allowing for main effects and interaction of shoes and runners (be specific on parameters and ranges of subscripts):

$$y_{ijk} = \mu + \alpha_i + b_j + (ab)_{ij} + e_{ijk}$$

$i = 1, 2, 3 \quad j = 1, \ldots, 4 \quad k = 1, \ldots, 5$

$\sum_i \alpha_i = 0$

$b_j \sim N(0, \sigma_b^2)$

$(ab)_{ij} \sim N(0, \sigma_{ab}^2)$

$e_{ijk} \sim N(0, \sigma^2)$

Indep.

(10)

p.2.b. Complete the following table. Whenever possible, use actual numbers, not symbols. For the sum of squares, write out the relevant summation.

Source	df (2 each)	Sum of Squares (5 each)	E{MS} (6 each)
Shoes (A)	$3-1=2$	$4(5)\sum_i(\bar{y}_{i..}-\bar{y}_{...})^2$	$\sigma^2 + 5\sigma_{ab}^2 + 4(5)\frac{\sum \alpha_i^2}{3-1}$
Runners (B)	$4-1=3$	$3(5)\sum_j(\bar{y}_{.j.}-\bar{y}_{...})^2$	$\sigma^2 + 5\sigma_{ab}^2 + 3(5)\sigma_b^2$
S*R	$2(3)=6$	$5\sum_i\sum_j(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2$	$\sigma^2 + 5\sigma_{ab}^2$
Error	$3(4)(5-1)=48$	$\sum\sum\sum(y_{ijk}-\bar{y}_{ij.})^2$	σ^2
TOTAL	$3(4)(5)-1=59$	$\sum_i\sum_j\sum_k(y_{ijk}-\bar{y}_{...})^2$	

(80)

Q.3. A wine tasting is conducted, with a sample of a judges, each rating a sample of b wines. Note that each judge only rates each wine once.

$$y_{ij} = \mu + a_i + b_j + e_{ij} \quad i = 1,\ldots,a;\ j = 1,\ldots,b \quad a_i \sim NID\left(0,\sigma_a^2\right) \quad b_j \sim NID\left(0,\sigma_b^2\right) \quad e_{ij} \sim NID\left(0,\sigma^2\right) \quad \{a_i\} \perp \{b_j\} \perp \{e_{ij}\}$$

p.3.a. Showing all work, obtain: $\overset{3}{E\{y_{ij}\}}, \overset{4}{V\{y_{ij}\}}, \overset{3}{E\{\bar{y}_{i\bullet}\}}, \overset{\cancel{8}0}{V\{\bar{y}_{i\bullet}\}}, \overset{3}{E\{\bar{y}_{\bullet j}\}}, \overset{\cancel{8}0}{V\{\bar{y}_{\bullet j}\}}, \overset{3}{E\{\bar{y}_{\bullet\bullet}\}}, \overset{\cancel{8}0}{V\{\bar{y}_{\bullet\bullet}\}}$

$E\{Y_{ij}\} = E\{\mu + a_i + b_j + e_{ij}\} = \mu + 0 + 0 + 0 = \mu \qquad V\{Y_{ij}\} = \sigma_a^2 + \sigma_b^2 + \sigma^2$

$Cov\{Y_{ij}, Y_{ij'}\} = \sigma_a^2 \quad j \neq j' \qquad Cov\{Y_{ij}, Y_{i'j}\} = \sigma_b^2 \quad i \neq i' \qquad Cov\{Y_{ij}, Y_{i'j'}\} = 0 \quad i \neq i',\ j \neq j'$

$E\{\bar{Y}_{i.}\} = E\left\{\frac{1}{b}\sum_j Y_{ij}\right\} = \frac{1}{b} b\mu = \mu$

$V\{\bar{Y}_{i.}\} = V\left\{\frac{1}{b}\sum_j Y_{ij}\right\} = \frac{1}{b^2}\left[\sum_j V\{Y_{ij}\} + 2\sum\sum_{j<j'} Cov\{Y_{ij}, Y_{ij'}\}\right]$

$= \frac{1}{b^2}\left[b(\sigma_a^2 + \sigma_b^2 + \sigma^2) + 2\frac{b(b-1)}{2}\sigma_a^2\right] = \frac{1}{b^2}\left[b^2\sigma_a^2 + b\sigma_b^2 + b\sigma^2\right]$

$E\{\bar{Y}_{.j}\} = E\left\{\frac{1}{a}\sum_i Y_{ij}\right\} = \frac{1}{a} a\mu = \mu$

$V\{\bar{Y}_{.j}\} = V\left\{\frac{1}{a}\sum_i Y_{ij}\right\} = \frac{1}{a^2}\left[\sum_i V\{Y_{ij}\} + 2\sum\sum_{i<i'} Cov\{Y_{ij}, Y_{i'j}\}\right]$

$= \frac{1}{a^2}\left\{a(\sigma_a^2 + \sigma_b^2 + \sigma^2) + 2\frac{a(a-1)}{2}\sigma_b^2\right\}$

$= \frac{1}{a^2}\left[a^2\sigma_b^2 + a\sigma_a^2 + a\sigma^2\right]$

$E\{\bar{Y}_{..}\} = E\left\{\frac{1}{a}\sum_i \bar{Y}_{i.}\right\} = \frac{1}{a} a\mu = \mu$

$V\{\bar{Y}_{..}\} = V\left\{\frac{1}{ab}\sum_i\sum_j Y_{ij}\right\} = \left(\frac{1}{ab}\right)^2 \Big\{\sum_i\sum_j V\{Y_{ij}\} + 2\sum\sum_{i<i'}\sum Cov\{Y_{ij}, Y_{i'j}\}$

$+ 2\sum\sum_{j<j'}\sum Cov\{Y_{ij}, Y_{ij'}\} + \sum\sum_{i\neq i'}\sum\sum_{j\neq j'} Cov\{Y_{ij}, Y_{i'j'}\}\Big\}$

$= \left(\frac{1}{ab}\right)^2\left\{ab(\sigma_a^2 + \sigma_b^2 + \sigma^2) + a(a-1)b\sigma_b^2 + ab(b-1)\sigma_a^2 + a(a-1)b(b-1)(0)\right\}$

$= \left(\frac{1}{ab}\right)^2\left\{a^2b\,\sigma_b^2 + ab^2\sigma_a^2 + ab\sigma^2\right\}$

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p.3.b. Derive $E\{MSA\} = E\left\{\frac{1}{a-1}\left[b\sum_{i=1}^{a}\bar{y}_{i\bullet}^{2} - ab\bar{y}_{\bullet\bullet}^{2}\right]\right\}$

$$E\{\bar{y}_{i.}^2\} = \mu^2 + \frac{1}{b^2}\left[b^2\sigma_a^2 + b\sigma_b^2 + b\sigma^2\right]$$

$$\Rightarrow E\{b\sum_i \bar{y}_{i.}^2\} = ab\left[\mu^2 + \sigma_a^2 + \frac{1}{b}\sigma_b^2 + \frac{1}{b}\sigma^2\right] \quad 10$$

$$E\{\bar{y}_{..}^2\} = \mu^2 + \frac{1}{b}\sigma_b^2 + \frac{1}{a}\sigma_a^2 + \frac{1}{ab}\sigma^2 \quad 10$$

$$\Rightarrow E\{ab\bar{y}_{..}^2\} = ab\left[\mu^2 + \frac{1}{b}\sigma_b^2 + \frac{1}{a}\sigma_a^2 + \frac{1}{ab}\sigma^2\right]$$

$$\Rightarrow E\{b\sum_i \bar{y}_{i.}^2 - ab\bar{y}_{..}^2\} = \mu^2(ab-ab) + \sigma_a^2(ab-b) + \sigma_b^2(a-a) + \sigma^2(a-1)$$

$$= b(a-1)\sigma_a^2 + (a-1)\sigma^2$$

$$\Rightarrow E\{MSA\} = \frac{1}{a-1}E\{SSA\} = \sigma^2 + b\sigma_a^2 \quad 10$$

p.3.c. The study had 9 judges and 10 varieties of wine. Complete the following ANOVA table testing:

$H_0^A: \sigma_A^2 = 0 \quad H_A^A: \sigma_A^2 > 0 \quad H_0^B: \sigma_B^2 = 0 \quad H_A^B: \sigma_B^2 > 0$

2 each   2 each   2 each   2 each

Source	df	SS	MS	F	F(0.05)
Judge	9-1=8	231.07	28.88	6.40	~2.074
Variety	10-1=9	23.89	2.65	0.59	~
Error	8(9)=72	324.98	4.51	—	—
Total	9(10)-1=89	579.95	—	—	—

p.3.d. Obtain unbiased estimates of σ_A^2 and σ^2.

$\tilde{\sigma}^2 = MSE = 4.51$ 4

$\tilde{\sigma}_A^2 = \frac{MSA - MSE}{b} = \frac{28.88 - 4.51}{10} = 2.437$ 6

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Q.4. An experiment was conducted to determine the effects of viewing a magical film versus a non-magical film in children. Samples of 32 6-year olds, and 32 8-year-olds were selected, and randomly assigned such that 16 of each age-group viewed the magical film and 16 of each age-group viewed the non-magical. The following table gives the means (SDs) for each treatment. The response (Y) was a score on an imagination scale (rating of a child acting out an object or animal).

Model: $y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \quad i=1,2 \; j=1,2 \; k=1,\dots,16 \quad \sum_{i=1}^{2}\alpha_i = \sum_{j=1}^{2}\beta_j = \sum_{i=1}^{2}(\alpha\beta)_{ij} = \sum_{j=1}^{2}(\alpha\beta)_{ij} = 0 \quad e_{ijk} \sim N(0,\sigma^2)$

p.1.a. The following table gives the means (SDs) for each treatment:

Age\Film	Non-Magical	Magical	Mean
6	17.0 (2.7)	21.6 (4.1)	19.3
8	18.7 (3.8)	22.7 (3.8)	20.7
Mean	17.85	22.15	20

$\hat{\alpha}_1 = -0.7 \quad \hat{\alpha}_2 = +0.7$

$\hat{\beta}_1 = -2.15 \quad \hat{\beta}_2 = +2.15$

$\widehat{(\alpha\beta)}_{11} = 17.0 - 19.3 - 17.85 + 20.0 = 0.15$

Complete the following ANOVA table:

$SSA: br\sum_i(\hat{\alpha}_i^2) = 2(16)[2(0.7^2)] = 31.36 \quad df_A = a-1 = 1 \quad MSA = 31.36$

$SSB = ar\sum_j \hat{\beta}_j^2 = 2(16)[2(2.15^2)] = 295.84 \quad df_B = b-1 = 1 \quad MSB = 295.84$

$SSAB = r\sum_i\sum_j \widehat{\alpha\beta}_{ij}^2 = 16[4(0.15^2)] = 1.44 \quad df_{AB} = (a-1)(b-1) = 1 \quad MSAB = 1.44$

$SSE = (r-1)\sum_i\sum_j s_{ij}^2 = (16-1)[2.7^2 + 4.1^2 + 3.8^2 + 3.8^2] = 794.7$

$df_E = 2(2)(16-1) = 60 \qquad MSE = 13.245$

Source	df	SS	MS	F	F(.05)
Age	1	31.36	31.36	2.368	4.001
Film	1	295.84	295.84	22.34	4.001
A*F	1	1.44	1.44	0.106	4.001
Error	60	794.7	13.245	#N/A	#N/A
Total	63	1123.34	#N/A	#N/A	#N/A

Q.5. An experiment is conducted to compare 4 navigation techniques (Factor A, Fixed), 2 input methods (Factor B, Fixed), in 36 subjects (Factor C, Random). Each subject is measured in each combination of levels of A and B once. Consider the model, where y is the task completion time:

$$y_{ijk} = \mu + \alpha_i + \beta_j + c_k + (\alpha\beta)_{ij} + (ac)_{ik} + (bc)_{jk} + e_{ijk} \quad \sum_{i=1}^{a}\alpha_i = \sum_{j=1}^{b}\beta_j = \sum_{i=1}^{a}(\alpha\beta)_{ij} = \sum_{j=1}^{b}(\alpha\beta)_{ij} = 0$$

$$c_k \sim NID(0,\sigma_c^2) \quad (ac)_{ik} \sim NID(0,\sigma_{ac}^2) \quad (bc)_{jk} \sim NID(0,\sigma_{bc}^2) \quad e_{ijk} \sim NID(0,\sigma^2) \quad \{c_k\} \perp \{(ac)_{ik}\} \perp \{(bc)_{jk}\} \perp \{e_{ijk}\}$$

Complete the following ANOVA table, testing all main effects and 2-way interactions.

$$F_c = \frac{2400.2 + 926.5}{1417.1 + 1960.1} = 0.985$$

Satterthwaite's Approximation: $W = \sum_i g_i MS_i \Rightarrow df_W \approx \dfrac{\left(\sum_i g_i MS_i\right)^2}{\sum_i \dfrac{(g_i MS_i)^2}{df_i}}$

$$df_{num} = \frac{3326.7^2}{\frac{2400.2^2}{35} + \frac{926.5^2}{105}} = 64.1$$

$$df_{Den} = \frac{3377.2^2}{\frac{1417.1^2}{105} + \frac{1960.1^2}{35}} = 88.5$$

ANOVA							
Source	df	SS	MS	F	df_num	df_den	F(0.05)
A	4-1=3	66996	22332	15.76	3	105	~2.691
B	2-1=1	30636	30636	15.63	1	35	~4.128
C	36-1=35	84008	2400.2	0.985	~64.1	~88.5	off chart
AB	3(1)=3	18710	6236.7	6.73	3	105	~2.691
AC	3(35)=105	148797	1417.1	1.53	105	105	off chart
BC	1(35)=35	68605	1960.1	2.12	35	105	off chart
Error	3(1)(35)=105	97282	926.5	—	—	—	—
Total	4(2)(36)-1 =287	515034	—	—	—	—	—

"Error Term"

$E\{MSA\} = \sigma^2 + 2\sigma_{ac}^2 + 2(36)\dfrac{\sum_i \alpha_i^2}{4-1}$ — MSAC

$E\{MSB\} = \sigma^2 + 4\sigma_{bc}^2 + 4(36)\dfrac{\sum_j \beta_j^2}{2-1}$ — MSBC

$E\{MSC\} = \sigma^2 + 4\sigma_{bc}^2 + 2\sigma_{ac}^2 + 4(2)\sigma_c^2$ — MSC + MSE vs MSAC + MSBC

$E\{MSAB\} = \sigma^2 + 36\dfrac{\sum_i\sum_j \alpha\beta_{ij}^2}{(4-1)(2-1)}$ — MSE

$E\{MSAC\} = \sigma^2 + 2\sigma_{ac}^2$ — MSE

$E\{MSBC\} = \sigma^2 + 4\sigma_{bc}^2$ — MSE

$E\{MSE\} = \sigma^2$ — —

Q.6. An unbalanced two-way ANOVA was conducted to compare ethics scores on an exam (Y) among members of the U.S. Coast Guard. The factors were Gender (X_1=1 if Male, -1 if Female) and Rank (X_2=1 if Officers, -1 if Enlisted). The sample sizes were: M/O = 72, M/E = 180, F/O = 15, F/E =32. Four regressions models were fit:

Model 1: $E\{Y_{ijk}\} = \beta_0 + \beta_1 X_{1ijk} + \beta_2 X_{2ijk} + \beta_3 X_{1ijk} X_{2ijk}$ $\quad SSE_1 = 42088$ $\quad \hat{Y}_1 = 36.35 - 2.40X_1 + 3.85X_2 - 0.50X_1X_2$

Model 2: $E\{Y_{ijk}\} = \beta_0 + \beta_1 X_{1ijk} + \beta_2 X_{2ijk}$ $\quad SSE_2 = 42122$ $\quad \hat{Y}_2 = 36.23 - 2.21X_1 + 3.52X_2$

Model 3: $E\{Y_{ijk}\} = \beta_0 + \beta_2 X_{2ijk} + \beta_3 X_{1ijk} X_{2ijk}$ $\quad SSE_3 = 42873$ $\quad \hat{Y}_3 = 34.75 + 3.30X_2 + 0.39X_1X_2$

p.6.a. Test whether there is an interaction between Gender and Rank. H_0: $\beta_3 = 0$ H_A: $\beta_3 \neq 0$ (2)

$$F_{obs} = \frac{\frac{42122 - 42088}{296 - 295}}{\frac{42088}{295}} = \frac{34}{142.67} = 0.24$$

(10)

(3) (2)

Test Statistic $F = 0.24$ Rejection Region $F \geq F_{.05,1,295} \approx 3.88$ P-value **> 0.05** or **< 0.05**

p.6.c. Test whether there is main effect for Gender. H_0: $\beta_1 = 0$ H_A: $\beta_1 \neq 0$ (2)

$$F_{obs} = \frac{\frac{42873 - 42088}{296 - 295}}{\frac{42088}{295}} = \frac{785}{142.67} = 5.50$$

(10)

(3) (2)

Test Statistic $F = 5.50$ Rejection Region $F \geq 3.88$ P-value **> 0.05** or **< 0.05**

p.6.c. Based on Model 2, give the predicted scores for all combinations of Gender and Rank.

Gender\Rank	Officer	Enlisted
Male	36.23-2.21+3.52 = 37.54	36.23-2.21-3.52 = 30.50
Female	36.23+2.21+3.52 = 41.96	36.23+2.21-3.52 = 34.92

4 each

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Q.7. A study was conducted, measuring the effects of 3 electronic **Readers** and 4 **Illumination** levels on time for people to read a given text (100s of seconds). There were a total of 60 subjects, 5 assigned to each combination of Reader/Illumination level. For this analysis, consider both Reader and Illumination level as fixed effects.

p.7.a Complete the following ANOVA table, and test for significant Reader/Illumination Interaction effects, as well as main effects for Reader and Illumination levels.

ANOVA	2 each		2 each	2 each	2 each	1 each
Source	**df**	**SS**	**MS**	**F**	**F(0.05)**	**Significant Effects?**
Reader	3-1=2	**70.70**	35.35	4.65	~3.183	Yes
Illumination	4-1=3	**148.11**	49.37	6.50	~2.790	Yes
Read*Illum	2(3)=6	**2.15**	0.36	0.05	~2.286	No
Error	3(4)(5-1)=48	**365.02**	7.60	—	—	
Total	3(4)(5)-1=59	**585.98**	—	—		

p.7.b Use Tukey's Method to make all pairwise comparisons among Readers.

$\overline{Y}_{1\bullet\bullet} = 1252.8 \quad \overline{Y}_{2\bullet\bullet} = 1032.2 \quad \overline{Y}_{3\bullet\bullet} = 1014.0$

$q(.05, 3, 48) = 3.420$ (4) $\quad \sqrt{\frac{MSE}{br}} = \sqrt{\frac{7.60}{4(5)}} = 0.616$ (5)

$HSD = 3.420(.616) = 2.11$ (2)

~~All are significantly different~~

$\underline{10.14 \quad 10.32} \quad 12.53$ (2)

p.7.c Use Bonferroni's method to make all pairwise comparisons among Illumination levels.

$\overline{Y}_{\bullet 1\bullet} = 1306.9 \quad \overline{Y}_{\bullet 2\bullet} = 1194.9 \quad \overline{Y}_{\bullet 3\bullet} = 970.7 \quad \overline{Y}_{\bullet 4\bullet} = 926.2$

$t(.025; 6, 48) \approx 2.747$ (4) $\quad \sqrt{\frac{2MSE}{ar}} = \sqrt{\frac{2(7.60)}{3(5)}}$ (5) $= 1.007$

$B_{ij} = 2.747(1.007) = 2.765$ (2)

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$\underline{9.262 \quad 9.707} \quad \underline{11.949 \quad 13.069}$ (2)