Chapter 14 - SPLIT-PLOT DESIGNS

SOME FACTORS DEMAND LARGE PLOTS, WHILE OTHER FACTORS MAY BE EASILY APPLIED TO SMALLER PLOTS.

LARGER TRT (WHOLE) PLOT IS SPLIT INTO SMALLER SUBPLOTS.
NEED RESTRICTIONS ON RANDOMIZATION.

EXAMPLE (Montgomery, 1991)

PAPER MANUFACTURER $y \equiv$ tensile strength of paper.

FACTOR: PULP PREPARATION METHOD (3 LEVELS)
FACTOR: Cooking Temperature (4 LEVELS)

$3 \times 4 = 12$ OBSERVATIONS PER REPLICATE RUN WANT 3 REPLICATES
PLANT CAN RUN 12 RUNS PER DAY => RUN ONE REPLICATE (BLOCK) PER DAY.

EXPERIMENT

① Large Batch of one pulp prep is produced, and broken into 4 samples, each randomly assigned to one of the 4 temperatures.
Repeat for the remaining pulp prep methods. Repeat across days.

② Note that this is not a randomized complete block w/ 2 trt factors, since we are not randomly assigning the 12 trt combinations to the 12 positions. Randomization Restriction

BLOCKS ARE BROKEN INTO 3 WHOLE PLOTS OF SIZE 4 UNITS (subplots)

WHOLE PLOT TRTS CONFOUNDED w/ WHOLE PLOTS; If possible make primary factor the split-plot (subplot) factor.

Let Factor $A \equiv$ Whole Plot Factor
" FACTOR $B \equiv$ Subplot factor

A Effects estimated from whole plots
B, AB EFFECTS ESTIMATED FROM subplots

$\rho \equiv$ Correlation between any observations on any 2 subplots in the same whole plot. Uncorrelated observations in different whole plots.

Error variance for main Effects of A on a per sub-plot basis is $\sigma^2[1+(b-1)\rho]$ If there are b subplots in each whole plot.

Error variance for main effects of B & AB interaction is $\sigma^2(1-\rho)$.

Experimental Error partitioned into 2 components: whole-plot & subplot

$$Y_{ijk} = \mu + \alpha_i + \rho_k + d_{ik} + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \qquad \begin{matrix} i=1,\dots,a \\ j=1,\dots,b \\ k=1,\dots,r \end{matrix}$$

μ = overall mean

$\alpha_i \equiv$ Effect of i^{th} level of factor A

$\rho_k \equiv$ " " k^{th} block — ρ_k could be r_k

$d_{ik} \equiv$ whole plot random error

$\beta_j \equiv$ Effect of j^{th} level of factor B

$(\alpha\beta)_{ij} \equiv$ Interaction of i^{th} level of A & j^{th} level of B.

$e_{ijk} \equiv$ Subplot random error

ANOVA

Source	df	MS	E[MS]	F_0
BLOCKS	$r-1$	MS BLOCKS		
A	$a-1$	MSA	$\sigma_e^2 + b\sigma_d^2 + rb\theta_a^2$	MSA/MSE(1)
Error (1)	$(a-1)(r-1)$	MSE(1)	$\sigma_e^2 + b\sigma_d^2$	
B	$(b-1)$	MSB	$\sigma_e^2 + ra\theta_b^2$	MSB/MSE(2)
AB	$(a-1)(b-1)$	MS(AB)	$\sigma_e^2 + r\theta_{ab}^2$	MS(AB)/MSE(2)
ERROR (2)	$a(r-1)(b-1)$	MSE(2)	σ_e^2	

NOTE: Assumes $\sigma_{br}^2 = \sigma_{abr}^2 = 0$

Alternative Formulation $\quad r \equiv$ BLOCKS

$$y_{ijk} = \mu + \alpha_i + r_k + (\alpha r)_{ik} + \beta_j + (\alpha\beta)_{ij} + (rb)_{jk} + (arb)_{ijk}$$

	F	F	R	R		
	a	b	r	1		
Factor	i	j	k	h	Component	E[MS]
α_i	0	b	r	1	θ_a^2	$\sigma^2 + \sigma_{arb}^2 + b\sigma_{ar}^2 + br\theta_a^2$
Whole plot: r_k	a	b	1	1	σ_r^2	$\sigma^2 + \sigma_{arb}^2 + a\sigma_{rb}^2 + b\sigma_{ar}^2 + ab\sigma_r^2$
$(ar)_{ik}$	1	b	1	1	σ_{ar}^2	$\sigma^2 + \sigma_{arb}^2 + b\sigma_{ar}^2$
β_j	a	0	r	1	θ_b^2	$\sigma^2 + \sigma_{arb}^2 + a\sigma_{rb}^2 + ar\theta_b^2$
$(\alpha\beta)_{ij}$	0	0	r	1	θ_{ab}^2	$\sigma^2 + \sigma_{arb}^2 + r\theta_{ab}^2$
$(rb)_{jk}$	a	1	1	1	σ_{rb}^2	$\sigma^2 + \sigma_{arb}^2 + a\sigma_{rb}^2$
$(arb)_{ijk}$	1	1	1	1	σ_{arb}^2	$\sigma^2 + \sigma_{arb}^2$
$e_{h(ijk)}$	1	1	1	1	σ_e^2	σ^2 (non-estimable)

Note: Kuehl's formulation assumes $\sigma_{rb}^2 = \sigma_{arb}^2 = 0$

$H_0: \sigma_{arb}^2 = 0$ UNTESTABLE

$H_0: \sigma_{rb}^2 = 0 \quad H_A: \sigma_{rb}^2 > 0 \quad$ T.S. $F_0 = \frac{MS(RB)}{MS(ARB)} = \frac{.4804}{.1260} = 3.81$

$F_{.05,2,6} = 5.14$

$H_0: \theta_a^2 = 0 \quad H_A: \theta_a^2 > 0 \quad$ T.S. $F_0 = \frac{MSA}{MS(AR)} = 29.62$

$H_0: \theta_b^2 = 0 \quad H_A: \theta_b^2 > 0 \quad$ T.S. $F_0 = \frac{MSB}{MS(RB)} = \frac{1.9079}{0.4804} = 3.97$

Or use combined $MS(RB)$, $MS(ARB)$

Example – Turfgrass (Kuehl)

i Factor A ≡ Nitrogen $(a=4)$
j Factor B ≡ Years $(b=3)$
k Blocks $(r=2)$

$\sum\sum\sum y_{ijk}^2 = 3.8^2 + \dots + 8.4^2 = 1017.79$ $\quad \sum\sum\sum y_{ijk} = 152.50$

$SS_{TOTAL} = 1017.79 - \frac{(152.50)^2}{24} = 48.78$ $\quad \bar{y}_{...} = 6.35$

$df_{TOTAL} = 24-1 = 23$

A Nit: $\bar{y}_{1..} = 4.77$ $\quad \bar{y}_{2..} = 5.75$ $\quad \bar{y}_{3..} = 6.77$ $\quad \bar{y}_{4..} = 8.13$

$SSA = 6\left[(4.77-6.35)^2 + (5.75-6.35)^2 + (6.77-6.35)^2 + (8.13-6.35)^2\right]$

$= 37.21$ $\quad MSA = \frac{37.21}{4-1} = 12.40$

B Years: $\bar{y}_{.1.} = 5.83$ $\quad \bar{y}_{.2.} = 6.45$ $\quad \bar{y}_{.3.} = 6.79$

$SSB = 8\left[(5.83-6.35)^2 + (6.45-6.35)^2 + (6.79-6.35)^2\right]$

$= 3.79$ $\quad MSB = \frac{3.79}{3-1} = 1.90$

Blocks: $\bar{y}_{..1} = 6.21$ $\quad \bar{y}_{..2} = 6.50$

$SS_{BLOCKS} = 12\left[(6.21-6.35)^2 + (6.50-6.35)^2\right] = 0.51$

$MS_{BLOCKS} = \frac{0.51}{2-1} = 0.51$

Error (1) ≡ Whole plot (nitrogen) × block interaction

~~SSE(1)~~ $\bar{y}_{1.1} = 5.0$ $\quad \bar{y}_{1.2} = 4.53$ $\quad \bar{y}_{2.1} = 5.40$ $\quad \bar{y}_{2.2} = 6.10$

$\bar{y}_{3.1} = 6.47$ $\quad \bar{y}_{3.2} = 7.07$ $\quad \bar{y}_{4.1} = 7.97$ $\quad \bar{y}_{4.2} = 8.30$

$SS_{ERROR(1)} = 3\left[(5.0-4.77-6.21+6.35)^2 + \dots + (8.30-8.13-6.50+6.35)^2\right]$

$= 3\left[0.14 + 0.15 + 0.04 + 0.04 + 0.03 + 0.02 + 0.00 + 0.00\right]$

$= 1.26$ $\quad MS_{ERROR(1)} = \frac{1.26}{(4-1)(2-1)} = 0.42$

A×B NIT*YEARS Interaction

$\bar{y}_{11.} = 3.85$ $\quad \bar{y}_{12.} = 5.35$ $\quad \bar{y}_{13.} = 5.10$

$\bar{y}_{21.} = 5.60$ $\quad \bar{y}_{22.} = 5.85$ $\quad \bar{y}_{23.} = 5.80$

$\bar{y}_{31.} = 6.50$ $\quad \bar{y}_{32.} = 6.00$ $\quad \bar{y}_{33.} = 7.80$

$\bar{y}_{41.} = 7.35$ $\quad \bar{y}_{42.} = 8.60$ $\quad \bar{y}_{43.} = 8.45$

$$SS(AB) = 2\left[(3.85-4.77-5.83+6.35)^2 + \cdots + (8.45-8.13-6.79+6.35)^2\right]$$

$$= 2\left[0.16+0.23+0.01+0.14+0.00+0.15+0.06+0.76+0.35 + 0.07+0.14+0.01\right] = 4.16$$

$$MS(AB) = \frac{4.16}{(4-1)(3-1)} = 0.69$$

ERROR(2) ≡ SUB PLOT * BLOCK INTERACTION + WP * SP * BLOCK INTERACTION

$$SS_{ERROR(2)} = SS_{TOTAL} - SSA - SSB - SSAB - SS_{BLOCKS} - SSE(1) =$$
$$= 48.78 - 37.21 - 3.79 - 4.16 - 0.51 - 1.26 = 1.85$$

COULD ALSO GET BY:

BLOCK * YEARS: $4\sum_j \sum_k (\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...})^2$

BLOCK × NIT × YEARS: $\sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij.} - \bar{y}_{i.k} - \bar{y}_{.jk} + \bar{y}_{i..} + \bar{y}_{.j.} + \bar{y}_{..k} - \bar{y}_{...})^2$

$$df_{ERROR(2)} = df_{TOTAL} - df_A - df_B - df_{AB} - df_{BLOCK} - df_{ERROR(1)}$$
$$= 23 - 3 - 2 - 6 - 1 - 3 = 8$$

COULD ALSO GET BY:

BLOCK * YEARS: $(2-1)(3-1) = 2$

BLOCK * NIT * YEARS: $(2-1)(4-1)(3-1) = 6$

$\boxed{2+6=8}$

ANOVA

	Source	df	SS	MS	F_0	$F_{\nu_1,\nu_2,.05}$
	BLOCKS	2-1=1	0.51	0.51	—	
Whole Plot	NITROGEN	4-1=3	37.21	12.40	12.40/0.42 = 29.52*	9.28
	ERROR (1)	(2-1)(4-1)=3	1.26	0.42	—	
	YEARS	3-1=2	3.79	1.90	1.90/0.23 = 8.26*	4.46
Sub Plot	NIT*YEAR	(4-1)(3-1)=6	4.16	0.69	0.69/0.23 = 3.00	3.58
	ERROR (2)	8	1.85	0.23	—	

ALTERNATIVE APPROACH – SEPARATES OUT YEAR*BLOCK & 3-WAY INTERACTION – SEE E(MS) ON P. 14.3

Source	df	SS	MS	F_0	$F_{.05,\nu_1,\nu_2}$
BLOCKS	1	0.51	0.51	—	
NITROGEN	3	37.21	12.40	29.52	9.28
ERROR (1)	3	1.26	0.42	—	
YEARS	2	3.79	1.90	1.90/0.48 = 3.96	19.00
BLOCKS*YEARS	2	0.96	0.48	0.48/0.13 = 3.69	5.14
NIT*YEAR	6	4.16	0.69	0.69/0.13 = 5.31*	4.28
BLOCKS*NIT*YEAR	6	0.76	0.13		

NOTE: Most Practicioners will combine BLOCK*SUBPLOT & 3-WAY INTERACTION AFTER FAILING TO REJECT THE NULL HYPOTHESIS THAT THE BLOCK×SUBPLOT VARIANCE COMPONENT IS 0, YIELDING MORE POWERFUL TESTS FOR SUBPLOT MAIN EFFECTS & INTERACTIONS. THIS IS KUEHL'S FORM. (TOP TABLE)

STANDARD ERRORS for TREATMENT FACTOR MEANS

Comparing 2 Levels of whole plot factor (A)

ESTIMATE: $\bar{Y}_{i..} - \bar{Y}_{j..}$ STD. ERROR: $\sqrt{\dfrac{2\,MSE(1)}{rb}}$

Comparing 2 Levels of Subplot factor (B)

ESTIMATE $\bar{Y}_{.u.} - \bar{Y}_{.v.}$ STD. ERROR: $\sqrt{\dfrac{2\,MSE(2)}{ra}}$

Comparing 2 Levels of B @ Same Level of A

ESTIMATE: $\bar{Y}_{iu.} - \bar{Y}_{iv.}$ STD. ERROR: $\sqrt{\dfrac{2\,MSE(2)}{r}}$

Comparing 2 Levels of A @ Same or different levels of B

Estimate: $\bar{Y}_{iu.} - \bar{Y}_{ju.}$ (Same B Level)

$\bar{Y}_{iu.} - \bar{Y}_{jv.}$ (Different B Level)

STD. ERROR*: $\sqrt{\dfrac{2[(b-1)MSE(2) + MSE(1)]}{rb}}$

* d.f. by Satterthwaite Approximation (Combining Mean Squares)

$$df = \frac{[(b-1)MSE(2) + MSE(1)]^2}{\dfrac{[(b-1)MSE(2)]^2}{df_{E(2)}} + \dfrac{[MSE(1)]^2}{df_{E(1)}}}$$

Turfgrass Example

Comparing Nitrogen (Whole Plot) Levels

STD. ERROR: $\sqrt{\frac{2\,MSE(1)}{rb}} = \sqrt{\frac{2(0.42)}{2(3)}} = 0.37$ w/ 3df

MINIMUM SIGNIFICANT DIFFERENCE (BONFERRONI) $k = \binom{4}{2} = 6$ comparisons

$t_{.025,6,3} \doteq t_{\frac{.05}{2(6)},3} = t_{.0042,3} \approx t_{.005,3} = 5.841$

MSD: $t_{.005,3} * \text{STD ERROR} = 5.841(0.37) = 2.16$

i, j	$\bar{y}_{i..} - \bar{y}_{j..}$
1,2	4.77 − 5.75 = −0.98
1,3	4.77 − 6.77 = −2.00
1,4	4.77 − 8.13 = −3.36*
2,3	5.75 − 6.77 = −1.02
2,4	5.75 − 8.13 = −2.38*
3,4	6.77 − 8.13 = −1.36

CONCLUDE ~~[illegible]~~

$NIT_4 > NIT_1$

$NIT_4 > NIT_2$

1 2 3 4

Comparing Year (subplot Levels)

STD. ERROR: $\sqrt{\frac{2\,MSE(2)}{ra}} = \sqrt{\frac{2(0.23)}{2(4)}} = 0.24$

Bonferroni table

$t_{.05,3,8} = 3.02$

MIN. SIG. DIFF. = 0.24(3.02) = 0.72

u, v	$\bar{y}_{.u.} - \bar{y}_{.v.}$
1,2	5.83 − 6.45 = −0.62
1,3	5.83 − 6.79 = −0.96*
2,3	6.45 − 6.79 = −0.34

CONCLUDE $YEAR_3 > YEAR\ 1$

1 2 3

COMPARING YEARS w/in NITROGEN TRTS (12 comparisons, 8 df)

STD. ERROR: $\sqrt{\frac{2\,MSE(2)}{r}} = \sqrt{\frac{2(0.23)}{2}} = 0.48$ $t_{.05,12,8} = 3.96$

MSD: 3.96 (.48) = 1.90

NIT 1

u,v	$\bar{y}_{1u.} - \bar{y}_{1v.}$
1,2	3.85 − 5.35 = −1.50
1,3	3.85 − 5.10 = −1.25
2,3	5.35 − 5.10 = 0.25

NIT 2

u,v	$\bar{y}_{2u.} - \bar{y}_{2v.}$
1,2	5.60 − 5.85 = −0.25
1,3	5.60 − 5.80 = −0.20
2,3	5.85 − 5.80 = 0.05

NIT 3

$\bar{y}_{3u.} - \bar{y}_{3v.}$
6.50 − 6.00 = 0.50
6.50 − 7.80 = −1.30
6.00 − 7.80 = −1.80

NIT 4

$\bar{y}_{4u.} - \bar{y}_{4v.}$
7.35 − 8.60 = −1.35
7.35 − 8.45 = −1.10
8.60 − 8.45 = 0.15

Comparing Nitrogens in common or different years

STD. ERROR: $\sqrt{\dfrac{2[(3-1)(0.23)+0.42]}{2(3)}} = 0.54$

$$df = \frac{(.88)^2}{\frac{(.46)^2}{8} + \frac{(.42)^2}{3}} = \frac{.7744}{.0853} = 9.08 \approx 9$$

Suppose want to compare all 6 pairs of NITROGEN TYPES IN ALL 3 YEARS OF THATCH => 18 pairwise comparisons

$t_{.05,18,9} \approx 4.07$

M.S.D. = 4.07 (0.54) = 2.20

i, j	YEAR 1: $\bar{y}_{i1.} - \bar{y}_{j1.}$	YEAR 2: $\bar{y}_{i2.} - \bar{y}_{j2.}$	YEAR 3: $\bar{y}_{i3.} - \bar{y}_{j3.}$
1, 2	3.85 - 5.60 = -1.75	5.35 - 5.85 = -0.50	5.10 - 5.80 = -0.70
1, 3	3.85 - 6.50 = -2.65*	5.35 - 6.00 = -0.65	5.10 - 7.80 = -2.70*
1, 4	3.85 - 7.35 = -3.50*	5.35 - 8.60 = -3.25*	5.10 - 8.45 = -3.35*
2, 3	5.60 - 6.50 = -1.10	5.85 - 6.00 = -0.15	5.80 - 7.80 = -2.00
2, 4	5.60 - 7.35 = -1.75	5.85 - 8.60 = -2.75*	5.80 - 8.45 = -2.65*
3, 4	6.50 - 7.35 = -0.85	6.00 - 8.60 = -2.60*	7.80 - 8.45 = -0.65

1 2 3 4   1 2 3 4   1 2 3 4