Chapter 9 – Introduction to Incomplete Block Designs

of trts $\equiv t > K \equiv$ # of experimental units / block

EXAMPLES –

1. 6 MACHINES ARE USED TO CREATE FINAL PRODUCT. BATCHES OF RAW MATERIAL HAVE ONLY ENOUGH MATERIAL FOR 4 ITEMS TO BE PRODUCED $(t=6, k=4)$

2. Movie. Production Company has 5 Scripts for potential films (actually called "treatments" in film biz). Have many readers, but only time enough for each reader to evaluate 2 Scripts. $(t=5, k=2)$

3. 7 varieties of fertilizer to be compared. Blocks of plots are only of size 4 (If you increase block size, lose homogeneity of units). $(t=7, k=4)$

BALANCED INCOMPLETE BLOCK DESIGN (BIBD)

Notation:
t = # of treatments
$k \equiv$ block size $(k < t)$
$r \equiv$ # of times each trt is replicated
$b \equiv$ # of blocks
$\lambda \equiv$ # of blocks any 2 pairs of trts occur together in

CONSTRAINTS

① Each trt paired w/ the $t-1$ remaining treatments λ times

$\Rightarrow \lambda(t-1)$ pairs per trt.

② Each trt appears in r blocks with $k-1$ other trts in each block. $\Rightarrow r(k-1)$ pairs/trt

$$\Rightarrow \boxed{\lambda(t-1) = r(k-1) \quad Ⓐ}$$

③ There are $N = bk$ measurements (b blocks, k units/blk)

④ " " $N = rt$ " (t trts, r reps/trt)

$$\Rightarrow \boxed{bk = rt \quad Ⓑ}$$

For BIBD, EACH EQUATION (Ⓐ and Ⓑ) must hold where all 6 "letters" being integers

EXAMPLE 1 : $t=6$ Machines, $k=4$ units/block

Ⓐ: $\lambda(6-1) = r(4-1) \Rightarrow 5\lambda = 3r$

Ⓑ $b(4) = r(6) \Rightarrow 4b = 6r$

$\Rightarrow 4b = 10\lambda \Rightarrow b = 2.5\lambda$

Minimum Design size

$t=6,\ k=4,\ r=10,\ b=15,\ \lambda=6$

See Plan 9A.3, Kuehl, p.330

for experimental design (Trt/block structure)

Also See Cochran and Cox (1957)

RANDOMIZATION FOR BIBD

1. RANDOMIZE ARRANGEMENT OF BLOCKS
2. RANDOMIZE ARRANGEMENT OF TRTS W/IN BLOCKS
3. RANDOMIZE ACTUAL TRTS TO TRT LABELS

EXAMPLE: $t=4$ BRANDS OF ANTIPERSPIRANT
$k=2$ armpits / subject

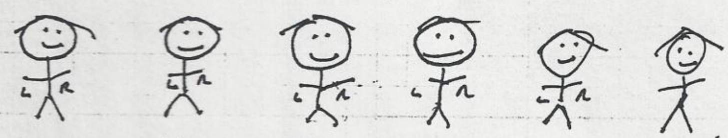
(A) $\lambda(4-1) = r(2-1) \Rightarrow 3\lambda = r$
(B) $b(2) = r(4) \Rightarrow 2b = 4r$

$\Rightarrow 12\lambda = 2b \Rightarrow b = 6\lambda$

$$\lambda = \frac{r(k-1)}{t-1} = \frac{r(1)}{3} \qquad r = 3 \Rightarrow \lambda = 1 \Rightarrow b = 6$$

TRTS: I $\boxed{\text{ARRID}}$ II $\boxed{\text{RG}}$ III $\boxed{\text{SURE}}$ IV $\boxed{\text{MENNEN}}$

BLOCKS:



Adam 1, Christian 2, Jie 3, Michel 4, Liu 5, MIN 6

$\boxed{\text{Subjects participate in 60 minutes of AEROBIC exercise}}$

POSITIONS IN BLOCKS: Left 1, Right 2

Pre-Randomization

BLOCK	POSITION LEFT	POSITION Right
Adam 1	1	2
christian 2	1	3
Jie 3	1	4
Michael 4	2	3
Liu 5	2	4
MIN 6	3	4

Randomizations based on 2 digit numbers from 3rd group, p. 627 (moving down column)

(1) Blocks: 80, 08, 13, 27, 48, 04

$\Rightarrow$ Adam gets 6th group
Christian gets 2nd group
Jie gets 3rd group
Micheal " 4th group
Liu " 5th group
MIN " 1st group

AMPAD 22-141 50 SHEETS 22-142 100 SHEETS 22-144 200 SHEETS

After 1^{st} Randomization

	Left	Right
MIN	1	2
Chris	1	3
Jie	1	4
Mike	2	3
Liu	2	4
Adam	3	4

After 2^{nd} RANDOMIZATION

	Left	Right
MIN	1	2
Chris	1	3
Jie	4	1
Mike	2	3
Liu	2	4
Adam	3	4

FINAL DESIGN

	L	R
MIN	A	R
Chris	A	M
Jie	S	A
Mike	R	M
Liu	R	S
Adam	M	S

4th group, P. 627 (moving down)

(2) TRTS w/in BLOCKS

BLOCK 1: 03, 80 $1 \to 2$

BLOCK 2: 32, 58 $1 \to 3$

BLOCK 3: 72, 34 $4 \to 1$

BLOCK 4: 03, 45 $2 \to 3$

BLOCK 5: 17, 33 $2 \to 4$

BLOCK 6: 48, ~~66~~ $3 \to 4$

5th group, P. 627, moving down

(3) Assign TRTS TO LABELS

TRTS: 07, 09, 53, 21

$\Rightarrow$ TRT 1 $\equiv$ ARRID (A)

TRT 2 $\equiv$ RG (R)

TRT 3 $\equiv$ MENNEN (M)

TRT 4 $\equiv$ SURE (S)

Analysis of BIBD (FIXED BLOCKS)

$$n_{ij}\, y_{ij} = n_{ij}(\mu + \tau_i + \beta_j + e_{ij}) \qquad \begin{matrix} i=1,\dots,t \\ j=1,\dots,b \end{matrix} \qquad n_{ij} = \begin{cases} 1 & \text{if trt } i \text{ appears in blk } j \\ 0 & \text{ow} \end{cases}$$

$$\boxed{\sum_i \tau_i = \sum_j \beta_j = 0 \qquad n_{i\cdot} = r \quad n_{\cdot j} = k}$$

$$Q = \sum_i \sum_j n_{ij}\, e_{ij}^2 = \sum_i \sum_j n_{ij}(y_{ij} - \mu - \tau_i - \beta_j)^2$$

$$\frac{\partial Q}{\partial \mu} = -2 \sum_i \sum_j n_{ij}(y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j) = 0$$

$$\Rightarrow \sum_i \sum_j n_{ij}\, y_{ij} = N\hat{\mu} + r \underbrace{\sum_i \hat{\tau}_i}_{(0)} + k \underbrace{\sum_j \hat{\beta}_j}_{(0)} = N\hat{\mu}$$

$$\Rightarrow y_{\cdot\cdot} = N\hat{\mu} \Rightarrow \hat{\mu} = \frac{y_{\cdot\cdot}}{N}$$

$$\frac{\partial Q}{\partial \tau_i} = -2 \sum_j n_{ij}(y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j) = 0$$

$$\Rightarrow y_{i\cdot} = r\hat{\mu} + r\hat{\tau}_i + \sum_j n_{ij}\hat{\beta}_j \qquad i = 1,\dots,t$$

$$\frac{\partial Q}{\partial \beta_j} = -2 \sum_i n_{ij}(y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j) = 0$$

$$\Rightarrow y_{\cdot j} = k\hat{\mu} + \sum_i n_{ij}\hat{\tau}_i + k\hat{\beta}_j \qquad j = 1,\dots,b$$

$$\Rightarrow \hat{\beta}_j = \frac{1}{k} y_{\cdot j} - \frac{1}{k} k\hat{\mu} - \frac{1}{k}\sum_i n_{ij}\hat{\tau}_i \qquad j=1,\dots,b$$

$$\left(k\hat{\beta}_j = y_{\cdot j} - k\hat{\mu} - \sum_i n_{ij}\hat{\tau}_i\right)$$

$$\Rightarrow k y_{i\cdot} = kr\hat{\mu} + kr\hat{\tau}_i + \sum_j n_{ij}\left[y_{\cdot j} - k\hat{\mu} - \sum_i n_{ij}\hat{\tau}_i\right]$$

$$i=1,\dots,t$$

Consider last term: $\sum_j n_{ij}\left[y_{\cdot j} - k\hat{\mu} - \sum_{i'} n_{i'j}\hat{\tau}_{i'}\right]$

① $\sum_j n_{ij} y_{\cdot j} \equiv$ sum of block totals for block containing Trt i $\equiv B_i$

② $\sum_j n_{ij} k\hat{\mu} = k\hat{\mu}\, n_{i\cdot} = kr\hat{\mu}$

③ $\sum_j n_{ij} \sum_{i'} n_{i'j}\hat{\tau}_{i'} = \sum_j n_{ij}^2 \hat{\tau}_i + \sum_{i'\neq i}\sum_j n_{i'j} n_{ij}\hat{\tau}_{i'}$

$n_{i'j} n_{ij} = 1$ if trts i and i' appear together in block j

TRTS i and i' appear together in λ blocks

$$\Rightarrow \sum_{i\neq i'}\sum_j n_{i'j} n_{ij}\hat{\tau}_{i'} = \sum_{i\neq i'} \hat{\tau}_{i'} \sum_j n_{i'j} n_{ij} = \sum_{i'\neq i} \hat{\tau}_{i'}\lambda$$

$$\Rightarrow \sum_j n_{ij} \sum_{i'} n_{i'j}\hat{\tau}_{i'} = r\hat{\tau}_i + \lambda \sum_{i'\neq i}\hat{\tau}_{i'}$$

Since $\sum_i \hat{\tau}_i = 0 \Rightarrow \hat{\tau}_i = -\sum_{i'\neq i}\hat{\tau}_{i'}$

$$= r\hat{\tau}_i - \lambda\hat{\tau}_i = (r-\lambda)\hat{\tau}_i$$

Putting together ①, ②, and ③, last term =

$$B_i - kr\hat{\mu} - (r-\lambda)\hat{\tau}_i$$

$$\Rightarrow k y_{i\cdot} = kr\hat{\mu} + kr\hat{\tau}_i + B_i - kr\hat{\mu} - (r-\lambda)\hat{\tau}_i$$

$$\Rightarrow k y_{i\cdot} - B_i = \hat{\tau}_i\left[kr - (r-\lambda)\right] = \hat{\tau}_i\left[r(k-1)+\lambda\right]$$

NOTE: $r(k-1) = \lambda(t-1)$ by definition

$$\Rightarrow k y_{i\cdot} - B_i = \hat{\tau}_i\left[\lambda(t-1)+\lambda\right] = \lambda t \Rightarrow \hat{\tau}_i = \frac{k y_{i\cdot} - B_i}{\lambda t} = \frac{kQ_i}{\lambda t}$$

where $Q_i = y_{i\cdot} - \frac{1}{k}B_i$

EXAMPLE: Tournament has 3 teams (TRTS), who play each of the other teams in 2 games each (games = Blocks). Teams are randomly assigned to labels (A, B, C). Games are spaced out so there is no carryover effect. Risk of player injury is 0. $t=3,\ r=4,\ k=2,\ \lambda=2,\ b=\frac{rt}{k}=6$

GAME (j)	TEAM (i) A	B	C	$y_{\cdot j}$	$\hat{\rho}_j^{\text{Full}}$	$\hat{\rho}_j^{\text{Reduced}}$
1	80		68	148	$8.8\bar{3}$	$10.\bar{6}$
2		48	56	104	$-9.1\bar{6}$	$-11.\bar{3}$
3	72	70		142	$7.\bar{3}$	$7.\bar{6}$
4		64	58	122	$-0.1\bar{6}$	$-2.\bar{3}$
5	76	58		134	$3.\bar{3}$	$3.\bar{6}$
6	52		58	110	$-10.1\bar{6}$	$-8.\bar{3}$
$y_{i\cdot}$	280	240	240	760		
B_i	534	502	484	$\hat{\mu}=\frac{760}{12}=63.\bar{3}$		
Q_i	13	-11	-2			
$\tilde{\tau}_i$	$\frac{13}{3}$	$-\frac{11}{3}$	$-\frac{2}{3}$			

NOTE: $B_1 = 148 + 142 + 134 + 110 = 534$

$$Q_1 = 280 - \tfrac{1}{2}(534) = 13$$

$$\tilde{\tau}_1 = \frac{2(13)}{2(3)} = \frac{13}{3}$$

$$\hat{\rho}_1^{\text{Full}} = \frac{1}{2}(148) - 63.\bar{3} - \frac{1}{2}\left[\frac{13}{3} + \left(-\frac{2}{3}\right)\right] = 8.8\bar{3}$$

$$\hat{\rho}_1^{\text{Reduced}} = \bar{y}_{\cdot 1} - \bar{y}_{\cdot\cdot} = \frac{148}{2} - 63.\bar{3} = 10.\bar{6}$$

Testing $H_0: \tau_1 = \tau_2 = \dots = \tau_t = 0$ H_A: NOT ALL τ_i ARE =

General Linear Test Approach

① UNDER FULL MODEL OBTAIN SSE_F

$$SSE_F = \sum_i \sum_j (y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\rho}_j^{Full})^2 \qquad \left(\hat{\rho}_j^{Full} = \bar{y}_{.j} - \bar{y}_{..} - \frac{1}{k}\sum_i n_{ij}\hat{\tau}_i\right)$$

② UNDER REDUCED MODEL (H_0) OBTAIN SSE_R

$$SSE_R = \sum_i \sum_j (y_{ij} - \hat{\mu} - \hat{\rho}_j^{REDUCED})^2 \qquad \left(\hat{\rho}_j^{Reduced} = \bar{y}_{.j} - \bar{y}_{..}\right)$$

③ Difference $\equiv SSE_R - SSE_F \equiv SS$(TRTS ADJUSTED FOR BLOCKS)

$$\equiv SST(\text{Adjusted}) = \frac{k\sum_i Q_i^2}{\lambda t}$$

Tournament example

TEAM i	GAME j	y_{ij}	$y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\rho}_F$	$(\cdot)^2$	$y_{ij} - \bar{y}_{.j}$	$(\cdot)^2$
1	1	80	$80 - 63.\bar{3} - \frac{13}{3} - 8.8\bar{3} = 3.5$	12.25	$80 - 74 = 6$	36
3	1	68	−3.50	12.25	−6	36
2	2	48	−2.50	6.25	−4	16
3	2	56	2.50	6.25	4	16
1	3	72	−3.00	9.00	1	1
2	3	70	3.00	9.00	−1	1
2	4	64	4.50	20.25	3	9
3	4	58	−4.50	20.25	−3	9
1	5	76	5.00	25.00	9	81
2	5	58	−5.00	25.00	−9	81
1	6	52	−5.50	30.25	−3	9
3	6	58	5.50	30.25	3	9
				$SSE_F = 206$		$SSE_R = 304$

$\Rightarrow SST(\text{Adjusted}) = 304 - 206 = 98$

NOTE: $\frac{k\sum_i Q_i^2}{\lambda t} = \frac{2}{2(3)}\left[13^2 + (-11)^2 + (-2)^2\right] = \frac{1}{3}(169 + 121 + 4) = \frac{294}{3} = 98$

~~ANALYSIS~~

- Reduced model: $df_E(R) = N-1-(b-1) = N-b$
- Full Model: $df_E(F) = N-1-(b-1)-(t-1) = N-b-t+1$

General Linear Test

$$F_0 = \frac{\dfrac{SSE(R)-SSE(F)}{df_E(R)-df_E(F)}}{\dfrac{SSE(F)}{df_E(F)}} = \frac{\dfrac{304-206}{[(12-6)-(12-6-2+1)]}}{\dfrac{206}{[12-6-2+1]}}$$

$$= \frac{98/1}{206/5} = 5\left(\frac{98}{206}\right) = \frac{490}{206} = 2.38$$

NOTE: the 1 df in numerator is more generally $\underline{t-1}$

RR: $F_0 \geq F_{.05,1,5} = 6.61$

Analysis of Variance

SOURCE	df	SS	MS
BLOCKS (UNADJUSTED)	$b-1$	$K\sum_j(\bar{Y}_{\cdot j}-\bar{Y}_{\cdot\cdot})^2$	$MSB(\text{unadj}) = \dfrac{SSB(\text{unadj})}{b-1}$
TRTS | BLOCKS (ADJUSTED)	$t-1$	$\dfrac{K\sum_i Q_i^2}{\lambda t}$	$MST(\text{adj}) = \dfrac{SST(\text{adj})}{t-1}$
ERROR	$N-b-t+1$	$\sum_i\sum_j(Y_{ij}-\hat{\mu}-\hat{\tau}_i-\hat{\rho}_j^F)^2$ or by subtraction	$MSE = \dfrac{SSE}{N-b-t+1}$
TOTAL	$N-1$	$\sum_i\sum_j n_{ij}(Y_{ij}-\bar{Y}_{\cdot\cdot})^2$	———

$H_0: \tau_1 = \cdots = \tau_t = 0$ $\quad H_A$: NOT ALL $\tau_i = 0$

T.S. $F_0 = \dfrac{MST(\text{adj})}{MSE}$

Under H_0, $F_0 \sim F_{t-1,\,N-t-b+1}$

Tournament Example

Source	df	SS
Games	5	778.67
Teams|Games	1	98
Error	5	206
TOTAL	11	1082.67

Inferences Concerning Means (Fixed Blocks)

$$\hat{\mu}_i = \hat{\mu} + \hat{\tau}_i = \frac{1}{rt} Y_{..} + \frac{k}{\lambda t} Y_{i.} - \frac{1}{\lambda t} B_i$$

$$Cov(Y_{ij}, Y_{i'j'}) = \begin{cases} \sigma^2 & \text{if } i=i', j=j' \\ 0 & \text{o.w.} \end{cases}$$

$V(Y_{..}) = rt\sigma^2$ (rt total measurements)

$V[Y_{i.}] = r\sigma^2$ (r reps/trt)

$V[B_i] = rk\sigma^2$ (r blocks/trt, k measurements/block)

$Cov(Y_{..}, Y_{i.}) = V(Y_{i.}) = r\sigma^2$ $\left(Y_{..} = Y_{i.} + \sum_{i \neq i'} Y_{i'.} \right)$

$Cov(Y_{..}, B_i) = V(B_i) = rk\sigma^2$

$Cov(Y_{i.}, B_i) = V[Y_{i.}] = r\sigma^2$

$$V[\hat{\mu}_i] = \overbrace{\left(\frac{1}{rt}\right)^2 [rt\sigma^2] + \left(\frac{k}{\lambda t}\right)^2 r\sigma^2 + \left(\frac{1}{\lambda t}\right)^2 rk\sigma^2}^{\text{VARIANCES}}$$

$$\underbrace{+ \frac{2k}{(rt)(\lambda t)} r\sigma^2 - \frac{2}{(rt)(\lambda t)} rk\sigma^2 - \frac{2k}{(\lambda t)^2} r\sigma^2}_{\text{Covariances}}$$

$$= \frac{\sigma^2}{rt}\left[1 + \frac{k^2 r^2 t}{(\lambda t)^2} + \frac{r^2 t k}{(\lambda t)^2} + \frac{2kr}{\lambda t} - \frac{2kr}{\lambda t} - \frac{2kr^2 t}{(\lambda t)^2}\right]$$

$$= \frac{\sigma^2}{rt}\left[1 + \frac{r^2k^2}{\lambda^2 t} + \frac{r^2 k}{\lambda^2 t} - \frac{2kr^2}{\lambda^2 t}\right] = \frac{\sigma^2}{rt}\left(1 + \frac{r^2k^2 - kr^2}{\lambda^2 t}\right)$$

$$= \frac{\sigma^2}{rt}\left[1 + \frac{r^2 k(k-1)}{\lambda t \left(\frac{r(k-1)}{t-1}\right)}\right] = \frac{\sigma^2}{rt}\left[1 + \frac{r^2 k(k-1)(t-1)}{\lambda t\, r(k-1)}\right]$$

$$\boxed{\lambda = \frac{r(k-1)}{t-1}} \qquad = \boxed{\frac{\sigma^2}{rt}\left[1 + \frac{rk(t-1)}{\lambda t}\right] = V(\hat{\mu}_i)}$$

$$\Rightarrow \boxed{S_{\hat{\mu}_i} = \sqrt{\frac{2MSE}{rt}\left[1 + \frac{rk(t-1)}{\lambda t}\right]}}$$

$$\hat{\mu}_i - \hat{\mu}_{i'} = \left(\frac{k}{\lambda t} y_{i\cdot} - \frac{1}{\lambda t} B_i\right) - \left(\frac{k}{\lambda t} y_{i'\cdot} - \frac{1}{\lambda t} B_{i'}\right)$$

$V[y_{i\cdot}] = V[y_{i'\cdot}] = r\sigma^2$ $\qquad Cov(y_{i\cdot}, B_i) = Cov(y_{i'\cdot}, B_{i'}) = r\sigma^2$

$V[B_i] = V[B_{i'}] = rk\sigma^2$

$cov(y_{i\cdot}, y_{i'\cdot}) = 0$

$Cov(y_{i\cdot}, B_{i'}) = \lambda\sigma^2$ (i appears in λ blocks w/ i')

$Cov(B_i, B_{i'}) = k\lambda\sigma^2$ (k observations/block in λ blocks)

$V(\hat{\mu}_i - \hat{\mu}_{i'}) =$

$$\left(\frac{k}{\lambda t}\right)^2 r\sigma^2 + \left(\frac{1}{\lambda t}\right)^2 rk\sigma^2 + \left(\frac{k}{\lambda t}\right)^2 r\sigma^2 + \left(\frac{1}{\lambda t}\right)^2 rk\sigma^2$$

$$-\frac{2k}{(\lambda t)^2} r\sigma^2 - 2\left(\frac{k}{\lambda t}\right)^2 (0) + \frac{2k}{(\lambda t)^2}\lambda\sigma^2 + \frac{2k}{(\lambda t)^2}\lambda\sigma^2 - 2\left(\frac{1}{\lambda t}\right)^2 k\lambda\sigma^2$$

$$-\frac{2k}{(\lambda t)} r\sigma^2$$

$$= \frac{\sigma^2}{(\lambda t)^2}\left[rk^2 + r\cancel{k} + rk^2 + rk - 2r\cancel{k} + \overset{\text{stays in}}{2\cancel{k}\cancel{\lambda}} + 2\cancel{k}\lambda - 2\cancel{k}\lambda - 2kr\right]$$

$$= \frac{2\sigma^2 k}{(\lambda t)^2}\left[rk - \lambda - r\right) = \frac{2k\sigma^2}{(\lambda t)^2}\left[r(k-1) + \lambda\right]$$

$\boxed{r(k-1) = \lambda(t-1)}$ $\qquad = \frac{2k\sigma^2}{(\lambda t)^2}\left[\lambda(t-1) + \lambda\right] = \boxed{\frac{2k\sigma^2}{\lambda t}}$

$$\Rightarrow \boxed{S_{\hat{\mu}_i - \hat{\mu}_{i'}} = \sqrt{\frac{2k\,MSE}{\lambda t}}}$$

$(1-\alpha)100\%$ CI for $\mu_i - \mu_{i'}$

$$(\hat{\mu}_i - \hat{\mu}_{i'}) \pm t_{\frac{\alpha}{2}, N-t-b+1} \cdot \sqrt{\frac{2k\,MSE}{\lambda t}}$$

where:
$$\hat{\mu}_i = \frac{1}{N} y_{..} + \frac{k}{\lambda t} y_{i.} - \frac{1}{\lambda t} B_i = \bar{y}_{..} + \frac{k Q_i}{\lambda t}$$

$$Q_i = y_{i.} - \frac{1}{k} B_i$$

Tournament Example ($c=3$ Comparisons)

$\hat{\mu}_1 = \hat{\mu} + \hat{\tau}_1 = 63.\bar{3} + 4.\bar{3} = 67.67$

$\hat{\mu}_2 = \hat{\mu} + \hat{\tau}_2 = 63.\bar{3} - 3.\bar{6} = 59.67$

$\hat{\mu}_3 = \hat{\mu} + \hat{\tau}_3 = 63.\bar{3} - 0.\bar{6} = 62.67$

$MSE = \frac{206}{5} = 41.2$

(see p. 9.9, General linear Test)

$\hat{\mu}_1 - \hat{\mu}_2 = 8.0 \quad \hat{\mu}_1 - \hat{\mu}_3 = 5.0 \quad \hat{\mu}_2 - \hat{\mu}_3 = -3.0$

$$S_{\hat{\mu}_i - \hat{\mu}_{i'}} = \sqrt{\frac{2k\,MSE}{\lambda t}} = \sqrt{\frac{2(2)(41.2)}{1(3)}} = 7.41$$

Bonferroni's t ($c=3$ comparisons) $t_{.025,3,5} = 3.53$

$$t_{.025,3,5} \cdot S_{\hat{\mu}_i - \hat{\mu}_{i'}} = 3.53(7.41) = 26.2$$

No means are close to significance

CASE OF RANDOM BLOCKS

Let $t=3,\ k=2,\ \lambda=1\ \ b=3,\ r=2$

	TRT 1	2	3	
Block 1	Y_{11}	Y_{21}		$Y_{.1}$
2	Y_{12}		Y_{32}	$Y_{.2}$
3		Y_{23}	Y_{33}	$Y_{.3}$
	$Y_{1.}$	$Y_{2.}$	$Y_{3.}$	$Y_{..}$

$V(Y_{ij}) = \sigma_r^2 + \sigma^2$

$Cov(Y_{11}, Y_{21}) = Cov(Y_{12}, Y_{32})$

$= Cov(Y_{23}, Y_{33}) = \sigma_r^2$

all other Covariances $= 0$

$B_1 = Y_{11} + Y_{21} + Y_{12} + Y_{32}$

$B_2 = Y_{11} + Y_{21} + Y_{23} + Y_{33}$

$B_3 = Y_{12} + Y_{32} + Y_{23} + Y_{33}$

~~$V[Y_{..}]$~~

	This case	General CASE
$V(Y_{..}) =$	$6[\sigma_r^2+\sigma^2] + 2(3)\sigma_r^2$	$rt[\sigma_r^2+\sigma^2] + 2b\frac{k(k-1)}{2}\sigma_r^2$
$V[Y_{i.}] =$	$2(\sigma_r^2+\sigma^2)$	$r[\sigma_r^2+\sigma^2]$
$V[B_i] =$	$4[\sigma_r^2+\sigma^2] + 4\sigma_r^2$	$rk[\sigma_r^2+\sigma^2] + 2r\frac{k(k-1)}{2}\sigma_r^2$
$Cov(Y_{i.}, Y_{i'.}) =$	σ_r^2	$\lambda\sigma_r^2$
$Cov(Y_{i.}, B_i) =$	$2(\sigma_r^2+\sigma^2) + 2\sigma_r^2$	$r(\sigma_r^2+\sigma^2) + r(k-1)\sigma_r^2$
$Cov(Y_{i.}, B_{i'}) =$	$(\sigma_r^2+\sigma^2) + \sigma_r^2$	$\lambda(\sigma_r^2+\sigma^2) + \lambda(k-1)\sigma_r^2$
$Cov(B_i, B_{i'}) =$	$2(\sigma_r^2+\sigma^2) + 2\sigma_r^2$	$\lambda k[\sigma_r^2+\sigma^2] + 2\lambda\frac{k(k-1)}{2}\sigma_r^2$

$$\hat{\mu}_i - \hat{\mu}_{i'} = \frac{K}{\lambda t} y_{i.} - \frac{1}{\lambda t} B_i - \frac{K}{\lambda t} y_{i'.} + \frac{1}{\lambda t} B_{i'}$$

$$V(\hat{\mu}_i - \hat{\mu}_{i'}) = \left(\frac{K}{\lambda t}\right)^2 r(\sigma_r^2 + \sigma^2) + \left(\frac{1}{\lambda t}\right)^2 \left[rk(\sigma_r^2 + \sigma^2) + rk(k-1)\sigma_r^2\right]$$

$$+ \left(\frac{K}{\lambda t}\right)^2 \left[r(\sigma_r^2 + \sigma^2)\right] + \left(\frac{1}{\lambda t}\right)^2 \left[rk(\sigma_r^2 + \sigma^2) + rk(k-1)\sigma_r^2\right]$$

$$- \frac{2K}{(\lambda t)^2} \left[r(\sigma_r^2 + \sigma^2) + r(k-1)\sigma_r^2\right]$$

$$- \frac{2k^2}{(\lambda t)^2} \left[\lambda \sigma_r^2\right] + \frac{2K}{(\lambda t)^2} \left[\lambda(\sigma_r^2 + \sigma^2) + \lambda(k-1)\sigma_r^2\right]$$

$$+ \frac{2K}{(\lambda t)^2} \left[\lambda(\sigma_r^2 + \sigma^2) + \lambda(k-1)\sigma_r^2\right] - \frac{2}{(\lambda t)^2} \left[\lambda k(\sigma_r^2 + \sigma^2) + \lambda k(k-1)\sigma_r^2\right]$$

$$- \frac{2k}{(\lambda t)^2} \left[r(\sigma_r^2 + \sigma^2) + r(k-1)\sigma_r^2\right]$$

$$= \left(\frac{1}{\lambda t}\right)^2 \Big[rk^2\sigma_r^2 + rk^2\sigma^2 + rk^2\sigma_r^2 + rk\sigma^2 + rk^2\sigma_r^2 + rk^2\sigma^2$$

$$+ rk^2\sigma_r^2 + rk\sigma^2 - 2rk^2\sigma_r^2 - 2rk\sigma^2 - 2k^2\lambda\sigma_r^2$$

$$+ 2k^2\lambda\sigma_r^2 + 2\lambda k\sigma^2 + 2k^2\lambda\sigma_r^2 + 2\lambda k\sigma^2 - 2\lambda k^2\sigma_r^2 - 2\lambda k\sigma^2$$

$$- 2rk^2\sigma_r^2 - 2kr\sigma^2 \Big]$$

BALANCED INCOMPLETE BLOCK DESIGN (Cochran & Cox, 1957 pp 447-9)

EXPERIMENT OF CORN HYBRIDS: y: YIELD OF CORN (POUNDS/PLANT) $t=13$ hybrids w/ $r=4$ reps/trt

@ $b=13$ locations w/ $k=4$ hybrids/location

[redacted] HYBRID (TRT) i

[redacted]

SITE (BLOCK) j	1	2	3	4	5	6	7	8	9	10	11	12	13	($y_{.j}$) BLOCK TOTAL	$\bar{y}_{.j}$
1	–	–	25.3	–	–	19.9	–	–	29.0	–	24.6	–	–	98.8	24.70
2	–	–	23.0	19.8	–	–	–	33.3	–	–	–	22.7	–	98.8	24.70
3	–	–	–	–	–	–	–	–	–	16.2	19.3	31.7	26.6	93.8	23.45
4	–	27.3	–	–	27.0	–	–	35.6	–	–	17.4	–	–	107.3	26.83
5	–	–	–	–	–	–	23.4	30.5	30.8	32.4	–	–	–	117.1	29.28
6	–	–	–	30.6	32.4	27.2	–	–	–	32.8	–	–	–	123.0	30.75
7	34.7	–	–	–	31.1	–	–	–	25.7	–	–	30.5	–	122.0	30.50
8	–	–	34.4	–	32.4	–	33.3	–	–	–	–	–	36.9	137.0	34.25
9	38.2	32.9	37.3	–	–	–	–	–	–	31.3	–	–	–	139.7	34.93
10	–	28.7	–	30.7	–	–	–	–	26.9	–	–	–	35.3	121.6	30.40
11	36.6	–	–	31.1	–	–	31.1	–	–	–	28.4	–	–	127.2	31.80
12	31.8	–	–	–	–	33.7	–	27.8	–	–	–	–	41.1	134.4	33.60
13	–	30.3	–	–	–	31.5	39.3	–	–	–	–	26.7	–	127.8	31.95
Row Total ($y_{i.}$)	141.3	119.2	120.0	112.2	122.9	112.3	127.1	127.2	112.4	112.7	89.7	111.6	139.9	1548.5	29.78
B_i	523.3	496.4	474.3	470.6	489.3	484.0	509.1	457.6	459.5	473.6	427.1	442.4	486.8		
Q_i	10.48	-4.90	1.43	-5.45	0.58	-8.70	-0.18	12.80	-2.48	-5.70	-17.08	1.00	18.20		

$B_i = \Sigma$ (Block totals for blocks containing trt i)

$Q_i = (\text{TRT TOTAL})_i - \frac{B_i}{k}$

$= y_{i.} - \frac{1}{k} \sum_j n_{ij}\, y_{.j}$

$n_{ij} = \begin{cases} 1 & \text{if trt } i \text{ in block } j \\ 0 & \text{o.w.} \end{cases}$

$t = 13, \; r = 4, \; b = 13, \; k = 4, \; \lambda = \frac{4(4-1)}{13-1} = 1$ CC2

$$SS_{TOTAL} = \sum_i \sum_j (y_{ij\cdot} - \bar{y}_{\cdot\cdot})^2 = \left[(25.3 - 29.78)^2 + (19.9 - 29.78)^2 + \dots + (39.3 - 29.78)^2 + (26.7 - 29.78)^2\right]$$

$$= 1556.15$$

$$SS_{BLOCKS\,(UNADJ)} = k \sum_j (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})^2 = 4\left[(24.70 - 29.78)^2 + \dots + (31.95 - 29.78)^2\right] = 4(172.36) = 689.44$$

$$SS_{TRTS\,(ADJ)} = \frac{k \sum_{i=1}^{t} Q_i^2}{\lambda t} = \frac{4}{1(13)}\left[(10.48)^2 + \dots + (18.20)^2\right] = \frac{4}{13}(1068.09) = 328.64$$

$$SS_{ERROR} = SS_{TOTAL} - SS_{BLOCKS\,(UNADJ)} - SS_{TRTS\,(ADJ)} = 1556.15 - 689.44 - 328.64 = 538.07$$

ANOVA

Source	df	SS	MS	F_0	$F_{.05,12,27}$
Blocks (UNADJ)	13-1=12	689.44	57.45	—	
TRTS (ADJ)	13-1=12	328.64	27.39	1.37	2.13
Error	13(4)-13-13+1=27	538.07	19.93	—	
TOTAL	13(4)-1=51	1556.15	—	—	

$H_0: \tau_1 = \tau_2 = \dots = \tau_{13} = 0$

H_A: Not all $\tau_i = 0$

T.S. $F_0 = \frac{MST(adj)}{MSE} = 1.37$

RR: $F_0 \geq F_{.05,12,27} = 2.13$

Estimate for mean for Trt i: $\hat{\mu}_i = \hat{\mu} + \hat{\tau}_i = \bar{y}_{\cdot\cdot} + \frac{kQ_i}{\lambda t}$

Standard Errors for TRT Means: $S_{\hat{\mu}_i} = \sqrt{\frac{MSE}{rt}\left(1 + \frac{kr(t-1)}{\lambda t}\right)} = 2.46$

STD. Errors for differences in Means: $S_{\hat{\mu}_i - \hat{\mu}_j} = \sqrt{\frac{2k\,MSE}{\lambda t}} = 3.50$